Power-Suppressed Thermal Effects from Heavy Particles

Eric Braaten
Yu Jia
The organization of this talk:

- Introduction
  - Review Matsumoto and Yoshimura's work.
  - Motivation to reconsider their model. Why Effective-Field-Theory approach?
- Bosonic model with pair annihilation.
  - Derive effective Lagrangian
  - Derive free energy density and energy density.
- Effective Hamiltonian
  - Some issues about Field-redefinition.
  - Calculate thermal average of Hamiltonian.
- Summary.
Introduction

Heavy particle immersed in the thermal environment of light particles.

Conventional wisdom:

\[ n_\phi = \left( \frac{MT}{2\pi} \right)^{3/2} e^{-M/T} \quad T \ll M \]

↑ Boltzmann-suppressed.

"New" contribution: (unstable heavy particle)

Off-shell effects: dominant in the tail \((w \sim T)\) of Breit-Wigner spectrum

\[ n_\phi = \int \frac{d^3p}{(2\pi)^3} \int_0^\infty dw \frac{\Gamma/2\pi}{(w - E(p))^2 + (\Gamma/2)^2} \frac{1}{e^{w/T} - 1} \]

\[ \approx \frac{\Gamma}{4\pi^3 M^2} \int_0^\infty dw \frac{1}{e^{w/T} - 1} \int_0^w dp \; p^2 \]

\[ = \frac{\pi}{180} \frac{\Gamma}{M^2} T^4 \quad \text{for} \quad T \ll \Gamma \ll M \]

↑ power-suppressed
When $T < \langle M$, the power-suppressed term can easily dominate over Boltzmann-suppressed term.

**Cosmological implication:**

- Much larger relic abundance of WIMP than conventional prediction.
- Prolong the freeze-out time.
- Put tighter constraint on the properties of the heavy particles which are candidate for cold dark matter.
Method #1: Integrate out light fields to derive quantum kinetic equation.

Influential functional method
Hartree-Fock approx.

very hard to follow.

Method #2:

Direct thermal-field-theory calculation.

pretty clear, but rather complicated.

Consider a model in which heavy particles are stable, can only pair annihilate into light particles.
prior to renormalization. The propagator in thermal medium $\Delta_{\phi,\chi}(y)$ is periodic in the Euclidean time $y$, with a period $\beta = 1/T$, hence giving the range of integration $0 < \beta$ as explicitly indicated. The Fourier transformed propagator has the well known form, $\Delta(\omega_n, p) \sim 1/(-\omega_n^2 + p^2 + M^2)$ with discrete $\omega_n = 2\pi i n/\beta$ ($n = 0, \pm 1, \pm 2, \cdots$). The other contributions from Fig.2 are exponentially suppressed by $e^{-M/T}$ with $M$ the heavy $\phi$ particle mass, the factor familiar in the conventional approach.

As usual, one rewrites eq.(11) using the Fourier transform. The resulting discrete energy sum over $\omega_n$ can be converted to a contour integral of this variable $z = \omega_n$, using the function $1/(e^{eta z} - 1)$. After some algebraic manipulation, one finds that to $O[\lambda^2]$

$$\rho^{(2)}_\phi \sim -\lambda^2 \int dk \, dk' \, dp \, dp' (2\pi)^3 \delta(p + p' + k + k') \frac{2\omega_p}{\omega_p + \omega_{p'} - \omega_k - \omega_{k'}} \left[ f_p f_{p'} (1 + \frac{1}{f_k}) (1 + \frac{1}{f_{k'}}) - (1 + f_p)(1 + f_{p'}) f_k f_{k'} \right]$$

$$+ 2 \frac{f_p f_{p'} (1 + \frac{1}{f_k}) f_{k'} - (1 + f_p)(1 + f_{p'}) f_k (1 + f_{k'})}{(\omega_p + \omega_{p'} - \omega_k + \omega_{k'})^2}$$

$$+ f_p f_{p'} f_k f_{k'} - (1 + f_p)(1 + f_{p'})(1 + f_k)(1 + f_{k'}) \frac{1}{(\omega_p + \omega_{p'} + \omega_k + \omega_{k'})^2}$$

$$+ 2 \frac{f_p (1 + f_{p'})(1 + \frac{1}{f_k}) (1 + f_{k'}) - (1 + f_p) f_{p'} f_k f_{k'}}{(\omega_p - \omega_{p'} - \omega_k - \omega_{k'})^2}$$

$$+ 2 \frac{f_p (1 + f_{p'})(1 + f_{k'})(1 + f_k) - (1 + f_p) f_{p'} (1 + f_{k'}) f_k}{(\omega_p - \omega_{p'} + \omega_k - \omega_{k'})^2} \right], \quad (12)$$

We dropped minor Boltzmann suppressed terms to obtain this result. A shorthand notation for the phase space integral $dk = d^3k/(2\pi)^3 2\omega_k$ was used here, and $f_{p, p'}$ are the occupation number for the heavy $\phi$ particle, while $f_{k, k'}$ are that for the light $\chi$ particle;

$$f_p = \frac{1}{e^{\sqrt{p^2 + M^2} / T} - 1}, \quad f_k = \frac{1}{e^{k/T} - 1}.$$  \hspace{1cm} (13)

A similar form to eq.(12) was derived for the proper self-energy in ref.[5]. For simplicity we assume that the $\chi$ mass $m_\chi \ll T$, and indeed take $m_\chi = 0$ here.

Terms containing $f_p$ or $f_{p'}$ in eq.(12) are Boltzmann suppressed by $e^{-M/T}$. Drop-
**Bosonic Pair-Annihilation Model:**

\[ L(x, \phi) = L_x + \frac{1}{2}(\partial \mu \phi)^2 - \frac{i}{2} M^2 \phi^2 - \frac{1}{4!} \lambda \phi^4 x^4. \]

where \( L_x = \frac{1}{2}(\partial \mu x)^2 - \frac{1}{4!} \lambda x x^4. \)

Define \( H_\phi = \frac{1}{2} \dot{\phi}^2 + \frac{i}{2} (D \phi)^2 + \frac{1}{2} M^2 \phi^2. \)

\[ H_x = \frac{1}{2} \dot{x}^2 + \frac{i}{2} (D x)^2 + \frac{1}{24} \lambda x x^4. \]

\[ H_{\text{int}} = \frac{1}{4} \lambda \phi^4 x^4. \]

Interpret energy density \( \rho \) as

\[ \rho = \langle \hat{H}_\phi \rangle = \frac{\text{tr}(H_\phi e^{-\beta H})}{\text{tr} e^{-\beta H}} - (T = 0 \text{ contribution}) \]

Besides zeroth-order result:

\[ \rho_\phi = M \left( \frac{M T}{2 \pi} \right)^2 e^{-M/T}. \]

They found the power-suppressed terms

\[ \delta \rho_\phi = \frac{1}{69120} \lambda^2 \frac{T^6}{M^2}, \quad CANCEL! \]

\[ \delta \rho_x = -\frac{1}{69120} \lambda^2 \frac{T^6}{M^2}. \]

\[ \delta \rho_{\text{int}} = -\frac{\pi^2}{64800} \lambda^2 \frac{T^3}{M^4}. \]
Influential functional method needs integrating out the light fields, thus get a highly nonlocal action of heavy fields.

Is this natural?

When $T \ll M$, we barely can "see" any heavy particles which are on shell, what we can feel are only light particles, so maybe integrating out heavy particles is more natural strategy.

$\Rightarrow$ low energy Effective Theory.
A quote from R. Shankar:

"... and in addition to their beauty, effective field theories are also very *effective* in answering certain questions that the more microscopic versions cannot."
At low $T$, the characteristic energy scale of $x$ is $T$, the resolution length is about $1/T$.

But $\Phi$ are highly virtual, with lifetime $1/\lambda$.

Since $1/T \gg 1/\lambda$.

The $x$ particle cannot probe the heavy loop very accurately.

Heavy lines shrink to points.
Construct low-energy EFT.

standard way — matching

e.g. for \( \text{Left} = \chi \phi^+ \phi \phi^- \phi^+ \phi^- \).

\[
\begin{align*}
2i \left( p_1^2 (-p_2 \cdot p_3 - p_2 \cdot p_4 + p_3 \cdot p_4) \\
+ p_2^2 (-p_1 \cdot p_3 - p_1 \cdot p_4 + p_3 \cdot p_4) \\
+ p_3^2 (p_1 \cdot p_2 - p_1 \cdot p_4 - p_2 \cdot p_4) \\
+ p_4^2 (p_1 \cdot p_2 - p_1 \cdot p_3 - p_2 \cdot p_3) \right)
\end{align*}
\]

Matching with full theory

\[\chi \text{ light}\]

\[\phi \text{ heavy}\]
We're lucky here, since interaction is $x^2 \phi^2$.

**Gaussian Functional Integral.**

$$e^{i \text{Sett}(x)} = \int D\phi \ e^{i \int d^nx \ L}$$

$$\text{Sett} = \int d^nx \ L_x + \frac{i}{2} \ln \det (-\partial^2 - M^2 - \frac{\lambda}{2} x^2 + i\varepsilon)$$

$$= \int d^nx \ L_x + \frac{i}{2} \ln \det (-\partial^2 - M^2 + i\varepsilon) + \sum_{n=1}^{\infty} \text{Seff}^{(n)}(x).$$

where $\text{Seff}^{(n)}(x) = -\frac{i\lambda^n}{2^{n+1} \ n} \ \text{tr} \left[ (-\partial^2-M^2+i\varepsilon)^{-n} x^2 \right].$

Equivalent to one-loop matching. Much easier.

**Derivative Expansion**

E.g. $\text{Seff}^{(2)}(x) = (\log \text{an infraredly)} X^2$

$$+ \frac{\lambda^2}{16 (4\pi)^2} \sum_{n=1}^{\infty} \frac{(-1)^n \ \pi! (n-1)!}{(2n+1)! \ M^{2n}} \int d^nx \ x^2 (\partial^2)^n x^2.$$

\[\text{Sett}^{(1)} \quad \text{Sett}^{(2)} \quad \text{Sett}^{(3)}\]
We then get the effective Lagrangian:

\[
\mathcal{L}_{\text{eff}} = \mathcal{L} - \frac{\lambda^2}{96(4\pi)^2 M^2} \chi^2 \partial^2 \chi^2 + \frac{\lambda^2}{(960)(4\pi)^2 M^4} \chi^2 (\partial^2 \chi)^2 \chi^2 - \frac{\lambda^3}{96(4\pi)^2 M^2} \chi^6 + \cdots
\]

↑ NEW! \sim O(\lambda^3)

Free-energy density:

\[
\langle \chi^2 \partial^2 \chi^2 \rangle : \quad \delta F = 0
\]

\[
\langle \chi^2 (\partial^2 \chi)^2 \rangle : \quad \delta F = \frac{1}{1024} \frac{16}{225} \frac{\lambda^2 T^4}{M^4} F_{\text{free}}
\]

\[
\langle \chi^6 \rangle : \quad \delta F = -\frac{1}{1024} \frac{25}{48 \pi^4} \frac{\lambda^3 T^2}{M^2} F_{\text{free}}
\]

where \( F_{\text{free}} = -\frac{\pi^2}{12} T^4 \) — free energy density of ideal gas of free spin-0 boson.

Using \( p = -T^2 \frac{\partial}{\partial T} (\frac{F}{T}) \):

\[
\delta p = \frac{1}{1024} (\frac{112}{675} \frac{\lambda^2 T^4}{M^4} \left( 1 - \frac{125}{144 \pi^4} \frac{\lambda^3 T^2}{M^2} + \cdots \right) F_{\text{free}}
\]

where \( F_{\text{free}} = \frac{\pi^2}{30} T^4 \).

Notice absence of \( O(\lambda^2 \frac{T^6}{M^6}) \) term.

Recall Cancellation of \( \delta p \phi \) and \( \delta p x \) in M&Y's paper!
Effective Hamiltonian

Trouble in using $\text{Eff} = \hbar \frac{\partial \text{Eff}}{\partial \chi} = \text{Eff}$.

Since \text{Eff} contains term like

$$\chi^2 (\partial^2 \chi) \chi = 4 \partial^2 \chi \partial^2 \chi \chi + 8 \chi \partial^2 \chi \partial^2 \chi + 4 \chi \partial^2 \chi \partial^2 \chi$$

Depend on second derivative of $\chi$.

However notice All the operators which contain $\partial^2 \chi$
don't make contribution to dynamics!

\[ \times \text{scattering amplitude} \propto \sum_{i=1}^{4} \frac{p_i^2}{s} \]

$\chi^2 \partial^2 \chi \chi$ 

$= 4 m^2 \to 0$

EASY see from the Feynman Rule:

$\partial^2 \chi \to -p^2 (\overrightarrow{p})$

? How can we deal with these non-dynamical yet annoying operators?
Use our degree of freedom to redefine field to remove these non-dynamical operators.

Equation of motion: \((\partial^2 + m^2) \chi = 0\)

\[ \delta L_{\text{free}} = -\delta \chi (\partial^2 + m^2) \chi. \]

\[ \Rightarrow G(\chi, \ldots) \partial^2 \chi \text{ replaced by } -m^2 G(\chi, \ldots) = 0 \]

We found under such field-redefinition:

\[ \chi \rightarrow \chi - \frac{\lambda^2}{72(4\pi)^2 M^2} \chi^3 + \frac{\lambda^2}{720(4\pi)^2 M^4} \partial^2 \chi^2 + \ldots \]

The three pieces of redundant operators could be eliminated.

New effective Lagrangian

\[ L_{\text{eff}} = \chi + \frac{\lambda^2}{240 (4\pi)^2 M^4} (\partial \chi \partial^2 \chi)^2 - \frac{\lambda^3}{96 (4\pi)^2 M^2} \chi^6 + \ldots \]

\[ \Rightarrow \text{Now we can directly use Noether prescription} \]

\[ H_{\text{eff}} = \partial \chi + \delta \text{pow.} \]

\[ \delta \text{pow} = \frac{\lambda^2}{240 (4\pi)^2 M^4} (\partial \chi \partial^2 \chi) \left( 3 \chi^2 + (\partial \chi)^2 \right) \]

\[ + \frac{\lambda^3}{96 (4\pi)^2 M^2} \chi^6 + \ldots \]
Use

$$\langle 0 \rangle_T = \frac{\text{Tr}(0 \ e^{-\beta H_T})}{\text{Tr} e^{-\beta H_4}} = \frac{1}{Z} \int \mathcal{D}x \ 0 \ e^{-\int_0^b \text{Left}(x) \ dx + \text{Right}(x)}$$

The subtlety is here we just assume this equation holds for effective theory. We cannot prove it.

$$\delta S_x = \frac{1}{1024} \left( \frac{16}{135} \lambda^2 \frac{T^4}{M^4} - \frac{25}{24} \lambda^2 \left( \frac{3}{32} \right) \right) \mathcal{P}_{\text{tree}}.$$  

$$\delta S_{\text{pow}} = \frac{1}{1024} \left( \frac{32}{675} \lambda^2 \frac{T^4}{M^4} + \frac{25}{144} \lambda^2 \left( \frac{7}{32} \right) \right) \mathcal{P}_{\text{tree}}.$$  

$$\delta S_x + \delta S_{\text{pow}}$$ reproduce our previous result of $S_P$ from free energy density calculation.
A Bonus.

Notice \( x^2 \partial x^+ = x^2 \partial^+ x^+ \approx x^2 \partial x \partial^+ x \).

It doesn't cause any trouble to get \( \text{Leff} \) from \( \text{Leff} \). Let's see what happens if we don't use field-redefinition to kill it.

\[
L_{\text{pow}} \rightarrow L_{\text{pow}} + \frac{\lambda^2}{24(4\pi)^2 M^2} x^2 (\dot{x}^2 + 3 (\dot{\phi} x)^2).
\]

\[
L_{\text{eff}} \rightarrow L_{\text{eff}} + \frac{\lambda^2}{24(4\pi)^2 M^2} x^2 \partial x \partial^+ x.
\]

Then our \( \delta p_x \), \( \delta p_{\text{pow}} \) **exactly** recover Matsumoto and Yoshimura's result:

\[
\delta p_p = \frac{1}{69120} \frac{\lambda^2 T^6}{M^2} \quad (= \text{our } \delta p_{\text{pow}})
\]

\[
\delta p_x = -\frac{1}{69120} \frac{\lambda^2 T^6}{M^2} \quad (= \text{our } \delta p_x)
\]

Since \( x^2 \partial^+ x^+ \) can be eliminated by a field redefinition, so neither of their \( \delta p \) can have any physical significance.
Another good example: Kong & Ravndal 1993

In QED, integrate out the electron field

$$\text{Leff} = -\frac{i}{4} (F_{\mu \nu})^2 + \frac{\alpha}{\pi \hbar \gamma m_e} F_{\mu \nu} \partial^2 F^{\mu \nu} \quad \text{Uehling term}$$

$$+ \frac{\alpha}{\pi \hbar \gamma m_e} \left[ (F_{\mu \nu} F^{\mu \nu})^2 + \frac{2}{4} (F_{\mu \nu} F^{\mu \nu})^2 \right] + \ldots \quad \text{Euler-Heisenberg term}.$$ 

The Uehling term can be eliminated by following field redefinition:

$$A^\mu \rightarrow A^\mu + \frac{\alpha}{3 \pi \hbar \gamma m_e} \partial^2 A^\mu + \ldots,$$

so the leading term contributing to energy density is of order $\frac{\alpha^2 T^4}{m_e^2}$, instead of order $\frac{\alpha T^6}{m_e^2}$, as naively expected.
Summary.

1. EIT approach affords an efficient way to understand the origin of power-suppressed terms. They arise from effective actions among light particles that are induced by integrating out heavy particles.

2. Matsumoto and Yoshimura incorrectly attribute $\delta \rho_p = \frac{1}{6\gamma_{120}} \lambda^2 \frac{T^6}{M^2}$ to the heavy particle energy density, and then treat them as nonrelativistic real particles:

$$n_p = \frac{P_p + \delta P_p}{M} = (\frac{MT}{2\pi})^{3/2} e^{-M/T} + \frac{1}{6\gamma_{120}} \lambda^2 \frac{T^6}{M^2}.$$ 

As we have seen, $O(\lambda^2 \frac{T^6}{M^2})$ to energy density is just an Artifact, and can be easily eliminated by field redefinition.
3. \( \langle H_p \rangle \) cannot literally be interpreted as the energy density whose contribution only come from heavy particles.

\( H_p \) mixed with \( x^2, \dot{x}^2, (\dot{x}^2), \chi^\ast \)

when renormalization.

Therefore \( H_p \) also creates light particles via loop diagrams that involve virtual heavy particles.

So, in Matsumoto and Yoshimura's work, only \( \delta P = \delta P_p + \delta P_x + \delta P_\text{int} \) has physical meaning, not just single piece.