Hydrodynamics of relativistic systems

with

broken continuous symmetries
heavy-ion collisions, hydro models

QCD $SU(2)_L \times SU(2)_R \rightarrow SU(2)$
superfluid neutron liquid (neutron stars)

\[ \text{ULID} \rightarrow \text{nothing} \]
Role of broken symmetries in hydrodynamics

$T \neq 0$, far from any phase transitions

typical relaxation time $T_{\text{relax}}$

$ \rho \rightarrow \infty \quad t_{\text{relax}} \rightarrow \infty \quad (\gg T_{\text{relax}})$

2 categories of hydro d.o.f:
- conserved densities
- phases of order parameters (Goldstones)

Example: superfluid $\text{He}_4$

\[ \begin{align*}
\rho & \\
\mathbf{j} & \quad \text{conserved densities (also in normal phase)} \\
\mathbf{c} (s) & \\
\mathbf{\psi} & \quad \text{phase} \quad \langle \psi \rangle = \psi_0 e^{i\phi}
\end{align*} \]
Hydro of He\(^4\)

\[ \vec{v}_n \rightarrow \vec{v}_s = \frac{1}{m} \vec{V} \Phi \]

Thermal equilibrium:

\[ T, \mu, \vec{v}_n, \vec{v}_s = \text{const} \quad (|\vec{v}_n - \vec{v}_s| < \text{V} \text{crit}) \]

Hydro:

\[ T, \mu, \vec{v}_n, \vec{v}_s \quad \text{slowly varying} \]
POISSON BRACKET METHOD

Nonrelativistic

\[ n = \psi^\dagger \psi \]
\[ \frac{\partial}{\partial t} \psi^\dagger \psi = mn \]

\[ \{ j_i (x), n(y) \} = n(x) \Theta; \delta(x-y) \]
\[ \{ j_i (x), j_k (y) \} = \left( j_i (x) \frac{\partial}{\partial x^i} - j_k (y) \frac{\partial}{\partial y^k} \right) \delta(x-y) \]

Hamiltonian:

\[ H = \int d^3 x \left( \frac{\partial^2}{2mn} + F(n) \right) \]

\[ \dot{\psi} = [H, \psi] \]
\[ \frac{\partial n}{\partial t} + \Theta; (nv^i) = 0 \]
\[ \frac{\partial}{\partial t} j_k + \Theta; \Pi_{ik} = 0 \]
\[ \Pi_{ik} = mn v_i v_k + \delta_{ik} \rho \]

 describes ideal hydro of isentropic \( \frac{s}{n} = \text{const} \) fluid
Relativistic superfluid

\[ L = \partial_{\mu} \phi^* \partial^{\mu} \phi - V(\phi) \]

Hydro variables:

\[ S \]
\[ n = -i \phi^* \partial_0 \phi \]
\[ T^{0i} = \partial^0 \phi \partial^i \phi \]
\[ \varphi \]

Poisson brackets

\[ \left[ T^{0i}(x), T^{0k}(y) \right] = \left( T^{0k}(x) \frac{\partial}{\partial x^i} - T^{0i}(y) \frac{\partial}{\partial y^k} \right) \delta(x-y) \]
\[ \left[ T^{0i}(x), n(y) \right] = n(x) \partial_i \delta(x-y) \]
\[ \left[ T^{0i}(x), s(y) \right] = s(x) \partial_i \delta(x-y) \]
\[ \left[ T^{0i}(x), \varphi(y) \right] = -\partial_i \varphi \delta(x-y) \]
\[ \left[ n(x), \varphi(y) \right] = -\delta(x-y) \]

{ postulated }
Hamiltonian:

\[ H = \int dx \ T^{00}(s, n, T^{0i}, \partial_i \psi) \]

\[ \uparrow \]

"equation of state"

Equations of motion

\[ \dot{A} = [H, A] \quad A = s, n, T^{0i}, \psi \]

Lorentz invariance (imposed by hand)

\[ \dot{T}^{00} + \partial_i T^{0i} = 0 \]

constrains the form of function \( T^{00} \)

\[ \downarrow \]

Lots of calculations

\[ \downarrow \]
Relativistic superfluid hydro: covariant formulation

- Equilibrium: defined by an equation of state
  \[ p = p(T, \mu, \frac{1}{2} (\Theta \mu \nu)^2) \]
  \[ dp = s dT + n d\mu + v^2 d \left( \frac{1}{2} (\Theta \mu \nu)^2 \right) \]
  \[ p = s T + n \mu - \rho \]

- Hydrodynamics:
  \[ \Theta \mu T^{\mu \nu} = 0 \]
  \[ \Theta \mu (n u^\mu - v^2 \Theta^\mu \varphi) = 0 \]
  \[ \Theta \mu (s u^\mu) = 0 \]
  \[ u^\mu \Theta^\mu \varphi + \mu = 0 \]

\[ T^{\mu \nu} = (\rho + p) u^\mu u^\nu - pg^{\mu \nu} + v^2 \delta^{\mu \nu} \psi \bar{\psi} \]

\[ v = 1 < \psi > \]

"Josephson equation"

equivalent (in nontrivial way) to formulation by Carter, Khalatnikov, Lebedev
Note:

\[ \partial_\mu \partial_\nu \phi \text{ small} \quad p = p(T, \mu, \frac{1}{2} (\partial_\mu \phi)^2) \]

If one requires \( \phi \) to be slowly varying

\[ \partial_\mu \phi \text{ small:} \]

\[ p = p_0(T, \mu) + v^2(T, \mu) \cdot \frac{1}{2} (\partial_\mu \phi)^2 \]

\( \Rightarrow \) two "equations of states"

\[ p = p_0(T, \mu) \]

\[ v^2 = v^2(T, \mu) \]
Nuclear matter near chiral limit

Hydro d.o.f.

\[ S \]

\[ T^{0i} \]

\[ n_B \]

\[ \rho_L^a = \bar{\psi} \delta^0 \frac{1-\delta^5}{2} \frac{\tau^a}{2} \psi \]

\[ \rho_R^a = \bar{\psi} \delta^0 \frac{1+\delta^5}{2} \frac{\tau^a}{2} \psi \]

\[ \Sigma \in SU(2) \quad \text{phases of } \langle \bar{\psi} \psi \rangle \]

additional P.B.:

current algebra

\[ [\rho_L^a(x), \rho_L^b(y)] = -f^{abc} \rho_L^c \delta(x-y) \]

\[ [\rho_L, \rho_R] = 0 \]

transformation properties of \( \Sigma \)

\[ [\rho_L^a, \Sigma] = -i \lambda^a \Sigma \]

\[ [\rho_R^a, \Sigma] = i \Sigma \lambda^a \]
Final equations:

\[ i \partial_\mu \left[ (f_t^2 - f_s^2) u^\mu u^\nu \Sigma \partial_\nu \Sigma^+ + f_s^2 \Sigma \partial^\mu \Sigma^+ \right] + \left[ u^\mu \Sigma \partial_\mu \Sigma^+, \alpha \right]_- = 0 \]

\[ \partial_\mu (\alpha u^\mu) + \frac{1}{2} \left[ u^\mu \Sigma \partial_\mu \Sigma^+, \alpha \right]_- = 0 \quad \alpha = \alpha^a \lambda^a \]

\[ \partial_\mu (n u^\mu) = 0 \]

\[ \partial_\mu T^{\mu\nu} = 0 \]

where:

\[ T^{\mu\nu} = (\rho + p) u^\mu u^\nu - p g^{\mu\nu} + \frac{f_s^2}{4} \text{tr} \left( \partial^\mu \Sigma \partial^{\mu} \Sigma^+ + \Sigma \partial^\mu \partial^{\mu} \Sigma^+ \right) \]

\[ p = p_0 + \frac{1}{4} (f_t^2 - f_s^2) u^\mu u^\nu \text{tr} \partial_\mu \Sigma \partial_\nu \Sigma^+ + \frac{f_s^2}{4} \text{tr} \partial^\mu \partial^\mu \Sigma^+ + \frac{1}{f_s^2} \text{tr} \alpha^2 \]

Thermodynamics determines

\[ p_0 = p_0(T, \mu) \]

\[ f_{t,s,v} = f_{t,s,v}(T, \mu) \]
Conclusion:

- If symmetries broken, hydro describes fluid dynamical + Goldstone modes

- Eqns derivable from Poisson brackets not manifestly relativistically covariant other methods?

- Applications:
  - Nuclear matter near chiral limit
  - DCC, $\pi$-phonon interaction....

- Relation to kinetic theory?

- Dissipation, transport coefficients?

- Near phase transition: more hydro modes