PRODUCT INTEGRALS

WILSON LINES & LOOPS

NON-ABELIAN STOKES THEOREMS

Robert L. Karp
Freydoon Mansouri

= UNIVERSITY OF CINCINNATI =
OUTLINE

1.) Motivation for PI
2.) Precise def. & existence
3.) Properties
4.) Wilson Lines & Loops
5.) NAST & gauge transf. properties
6.) SUSY extension.

+ Conf. proc.
Traces go to Volterra:

"It is to the product what ordinary I is to the sum.

Problem: \( Y'(s) = A(s) Y(s) : Y(s_0) = Y_0 \in \mathbb{C}^n \)

\[ A \in \text{M}_{m \times n}(\mathbb{C}) \]

Approximate sol.: \( I = \{ s_0, \ldots, s_k \} \)

\[ s_0 = a \quad \text{[and]} \quad s_k = b \]

\[ \Delta s_i = s_i - s_{i-1} \quad i = 0, k \]

\[ Y(s_i) \approx e^{A(s_i) \Delta s_i} Y(a) \]

\[ Y(s_2) \approx e^{A(s_2) \Delta s_2} e^{A(s_1) \Delta s_1} Y(a) \]

\[ Y(s_k) = Y(b) \approx e^{A(b) \Delta s_k} \ldots e^{A(s_i) \Delta s_i} Y(a) \]

Q: Converges if \( \| A \| \| s \| \to 0 \) ?

If yes:

\[ Y(b) = \left( \sum_{i=0}^{k} e^{A(s_i) \Delta s_i} \right) Y(a) \]
**Precise Formulation**

**Def:** \( A: [a, b] \rightarrow \text{Max}_n(C) \quad P = \{s_0, \ldots, s_n\} \) part.

1. \( A \) is a step fn. \iff \exists P \text{ part. :} 
   \[ A\mid_{(s_i, s_{i+1})} = \text{const} \quad \forall i \in \mathbb{N} \]

2. \( A_P \) the pointwise approximation
   corresponding to \( A \& P \):
   \( A_P(s) = A(s_k) \) on \( (s_k, s_{k+1}) \)
   \( \text{step fn.} \quad A_P(s) = A(s_k) \)

3. For \( A \) step fn. define \( E_A: [a, b] \rightarrow \mathbb{R} \)
   \[ E_A(x) = e^{A(s_k)(x-s_{k-1})} \cdots e^{A(s_1)\Delta s_i} \]
   \( \forall x \in (s_0, s_n) \)

**Def - Thm**

If \( A: [a, b] \rightarrow \text{Max}_n(C) \) cont., \{ \( A_n \) \} \subset \( L^1([a, b]) \)

of step funs.: \( A_n \xrightarrow{L^1([a, b])} A \)

\( \{ E_{A_n} \} \) conv. uniformly to a matrix

\[
\begin{bmatrix}
E_{A_n}
\end{bmatrix}
\xrightarrow{\text{conv.}}
\begin{bmatrix}
a & b
\end{bmatrix}
\]
PROPERTIES OF PI'S

1) Integral - differential eq. \[ F(x,a) := \prod_{a}^{x} e^{A(s)}ds \]
\[
\frac{d}{dx} F(x,a) = A(x) F(x,a) \quad F(x,a) = I_{xxm} \quad A \text{ cont.}
\]
\[
F(x,a) = 1 + \int_{a}^{x} ds \ A(s) \ F(s,a)
\]

2) \[
det \left( \prod_{a}^{x} e^{A(s)}ds \right) = e^{\int_{a}^{x} ds \ \text{Tr} \ A(s)}
\]

3) If \( [A(s), A(s')] = 0 \) then \[
\prod_{a}^{x} e^{A(s)}ds = e^{\int_{a}^{x} ds \ A(s)}
\]

4) \[
\prod_{a}^{b} e^{A(s)}ds = \prod_{c}^{b} e^{A(s)}ds \prod_{c}^{a} e^{A(s)}ds
\]

Consequence: \[
\left( \prod_{a}^{b} e^{A(s)}ds \right)^{-1} = \prod_{b}^{a} e^{A(s)}ds
\]

5) \[
\frac{\partial}{\partial x} \left( \prod_{y}^{x} e^{A(s)}ds \right) = A(x) \left( \prod_{y}^{x} e^{A(s)}ds \right)
\]
\[
\frac{\partial}{\partial y} \left( \prod_{y}^{x} e^{A(s)}ds \right) = -\left( \prod_{y}^{x} e^{A(s)}ds \right) A(y)
\]

6) "Sum Rule" \[
\prod_{a}^{b} e^{\left[ A(s) + B(s) \right]ds} = B(c) \prod_{a}^{x} e^{\prod_{a}^{b} B(s)P(s) \ ds}
\]
\[
P_{\text{sum}} = \prod_{a}^{x} e^{A(s)}ds
\]
7.) SIMILARITY RULE

\[ P(x) \left( \prod_a e^{B(s) ds} \right) P^{-1}(a) = \prod_a e^{\left[ LP(s) + P(s) B(s) P^{-1}(s) \right] ds} \]

where \( L \) - derivative

\[ (L A)(x) = A'(x) A^{-1}(x) \]

A rew. & diff.

8.) "NEWTON's LAW"

\[ \prod_a e^{LP(s) ds} = P(x) P^{-1}(a). \]

WILSON LINES & LOOPS

\[ M^n \text{ manifold} \]

\[ A^a_\mu \text{ connection, gauge field} \]

\[ \text{gauge group } G \text{, generators } T^a \in \mathfrak{g} \]

\[ A(x) = A^a_\mu (x) T^a dx^\mu \]
Let \( c : [a,b] \rightarrow M \) be a curve

\[ \mathcal{W}[c] := \mathcal{P} \in c^*A \]

i) \( \mathcal{P} \) open \( \Rightarrow \) Wilson line

ii) \( \mathcal{P} \) closed \( \Rightarrow \) Wilson loop

\[ \mathcal{P}^*A(s) = A_{\mu}(x(s)) \frac{dx^\mu}{ds}(s) \quad x^\mu = x^\mu(s) \quad \text{param.} \]

For simplicity: \( \mathcal{P}^*A \equiv A : [a,b] \rightarrow M_{\times n}(\mathbb{C}) \)

Define:

\[ \mathcal{P} \in c^*A \equiv \mathcal{P} \in c^*A \equiv \prod_{a}^{b} e^{-A(s)ds} \]

Goal: find a rep. of \( \mathcal{W}[c] \) involving the

\[ F = dA + A \wedge A \]

and \( S \) bounded by \( \mathcal{P} \) (\( c \) closed)
Local parameters on $S$: $s^a \begin{cases} s^t = 0 \\ s^b = z \end{cases}$

$$A_a = \pi_a^\mu A_\mu$$

$$F_{ab} = \pi_a^\mu \pi_b^\nu F_{\mu\nu}$$

$$s(t) = \partial_x x^\mu$$

$$\pi(s, s_0; t) := \prod_{s_0}^{s} e^{A_1(s', t) ds'}$$

$$Q(s; t, t_0) := \prod_{t_0}^{t} e^{A_0(s, t') dt'}$$

$$W[C] = Q^{-1}(s_0 ; t, z_0) P^{-1}(s, s_0; z) Q(s ; t, z_0) P(t, s_0; z)$$
THEOREM (NON-ABELIAN STOKES)

\[ W[C] = \prod_{z_0} e^{\int_{z_0}^{z} T^{-1}(z', z) F_{01}(z', z) T(z', z') \, dz'} \]

\[ T(z, \bar{z}) = P(z, z_0; \bar{z}) Q(z_0; \bar{z}, z_0) \]

Note: \( z \& \bar{z} \) play differently

Proof: \( z \) independent

Q: How do we know NAST consistent?

Useful check & illustration of the methods
THE GAUGE TRANSFORMS OF WILSON'S

\[ A_{\mu}(x) \rightarrow g(x) A_{\mu}(x) g^{-1}(x) - g(x) \partial_{\mu} g^{-1}(x) \]

\[ F_{\mu\nu}(x) \rightarrow g(x) F_{\mu\nu} g(x)^{-1} \]

\[ P(\vec{\ell}, \vec{e}_0, \vec{z}) = \prod_0^5 \int \frac{d^4 \ell}{d^3 0} e^{i \vec{\ell} \cdot \vec{e}_0} A_{1}(\vec{\ell}, \vec{z}) A_{1}(\vec{\ell}, \vec{z}) g(\vec{\ell}, \vec{z}) g^{-1}(\vec{\ell}, \vec{z}) g(\vec{\ell}, \vec{z}) \partial \vec{\ell} \]

\[ = q(\vec{\ell}, \vec{z}) P(\vec{\ell}, \vec{e}_0, \vec{z}) g^{-1}(\vec{\ell}, \vec{z}) \]

\[ \Rightarrow W[C] \rightarrow q(\vec{e}_0, \vec{z}_0) W[C] q^{-1}(\vec{e}_0, \vec{z}_0) \]

Q: what about the NAST form?

\[ Q(\vec{e}_0, \vec{z}_0) \rightarrow q(\vec{e}_0, \vec{z}_0) Q(\vec{e}_0, \vec{z}_0) g^{-1}(\vec{e}_0, \vec{z}_0) \]

\[ T(\vec{e}, \vec{z}) = P(\vec{e}, \vec{e}_0, \vec{z}) Q(\vec{e}_0, \vec{z}_0) \rightarrow q(\vec{e}, \vec{z}) T(\vec{e}, \vec{z}) g^{-1}(\vec{e}_0, \vec{z}_0) \]
\[ T(6, z) \rightarrow g(6, z) T(6, z) g^{-1}(6, z) \]

\[ W[C] = \prod_{\infty} e^{\left( \int_{6, \infty}^6 d\tilde{z} \left( f(6, \tilde{z}) T^{-1}(6, \tilde{z}) F_0(6, \tilde{z}) T(6, \tilde{z}) \right) \right) d\tilde{z}} \]

Once again: \[ W[C] \rightarrow g(6, \infty) W[C] g(6, \infty)^{-1}. \]

In addition

All/more of these results have been extended to the SUSY case and it does work.