LIGHT QUARK MASSES
FROM LATTICE QCD
AND SUM RULES
(DPF 2000, Columbus, OHIO)

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LIGHT QUARK MASSES

Light quark masses are poorly determined.
Review of Particle Properties by PDG (\(\overline{MS}; \mu = 2\) GeV)

<table>
<thead>
<tr>
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<th>1996</th>
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Direct determination of quark mass not possible:

- Quarks are not asymptotic states.
- \(m_{\text{constituent}} \sim \Lambda_{\text{QCD}}\).

Three approaches are used to extract these masses:

1. Chiral Perturbation Theory
2. Lattice QCD
3. Sum Rules
CHIRAL PERTURBATION THEORY

- Lowest order chiral lagrangian (same symmetries as QCD):

\[ \mathcal{L} = \frac{1}{4} f^2 \text{Tr} \partial_\mu \Sigma \partial^\mu \Sigma^\dagger + \frac{1}{2} f^2 \mu \text{Tr}(M\Sigma + M\Sigma^\dagger) \]

with unknown decay constant \( f \) and overall scale \( \mu \). So only ratios of masses predictable:

- Lowest order results:
  \[ \frac{m_s}{m} = 24.2-25.9 \]
  \[ \frac{m_u}{m_d} = 0.55 \]
  \[ \frac{m_s}{m_d} = 20.1 \]

- Naïve analysis does not change much at next order:
  \[ \frac{m_s}{m} = 24.4(1.5) \]
  \[ \frac{m_u}{m_d} = 0.553(43) \]
  \[ \frac{m_s}{m_d} = 18.9(8) \]
• Corrections to lowest order

\[ \frac{M_K^2}{M^2_{\pi}} = \frac{m_s + \bar{m}}{m_u + m_d} (1 + \Delta_M) \]
\[ \Delta_M > 0 \]

• \(O(m)\) corrections absent in the combination:

\[ Q \equiv \frac{M_K^2}{M^2_{\pi}} \frac{M_K^2 - M^2_{\pi}}{M_{K^0}^2 - M_{K^+}^2} = \frac{m_s^2 - \bar{m}^2}{m_d^2 - m_u^2} (1 + O(m^2)) \]
\[ = 22.7(8) \ (\eta \text{ decay}) \]

• A second independent ratio \(R\) (determined in various ways)

\[ R \equiv \frac{m_s - \bar{m}}{m_d - m_u} < 44 \]
Figure from Leutwyler (hep-ph/9609467). The dot is Weinberg’s value and the cross is estimate reported in Gasser and Leutwyler, Phys. Rep. 87 (1982) 77.
DETERMINATION OF $\Delta_M$?

- Kaplan Manohar symmetry: Chiral Lagrangian invariant under

$$
M' \rightarrow \alpha M + \beta (M^\dagger)^{-1} \text{det} M \\
m'_u \rightarrow \alpha m_u + \beta m_d m_s
$$

if LEC are suitably modified.

- KM symmetry implies that only $Q$ is determined without ambiguity at next to leading order.

- Hard to push chiral perturbation theory further. Leutwyler makes "reasonable" assumptions to extract two ratios from PS phenomenology.

- Need to include electromagnetic corrections to determine $m_u$ and $m_d$ separately.

- $m_u = 0$?
DEFINITION OF QUARK MASSES

Ward identities used to define quark masses:

1. Vector Ward Identity (VWI):

\[
\partial_\mu (V_\mu^{(12)})_R = (m_1 - m_2)_R(\mu)S_R^{(12)}(\mu) + O(a^{n+1})
\]

\[
\partial_\mu (ZV V_\mu^{(12)}) = Z_m(\mu a)(m_1 - m_2)(\frac{1}{a})Z_S(\mu a)S^{(12)}(\frac{1}{a})
\]

Introduces \(\kappa_c\) for Wilson like fermions that defines the chiral limit.

2. Axial vector Ward Identity:

\[
\partial_\mu (A_\mu^{(12)})_R = (m_1 + m_2)_R(\mu)P_R(\mu) + O(a^{n+1})
\]

\[
\partial_\mu (Z_A A_\mu^{(12)}) = Z_m(\mu a)(m_1 + m_2)(\frac{1}{a})Z_P(\mu a)P(\frac{1}{a})
\]

using which

\[
(m_1 + m_2)_R = \frac{Z_A}{Z_P(\mu)} \frac{\langle 0 | \partial_4 A_4(t)J(0) | 0 \rangle}{\langle 0 | P(t)J(0) | 0 \rangle}
\]

\[
t \to \infty \quad -m_\pi \frac{Z_A}{Z_P(\mu)} \frac{\langle 0 | A_4(t)J(0) | 0 \rangle}{\langle 0 | P(t)J(0) | 0 \rangle}
\]
RGI MASS

- Nature provides probes of axial ($A_\mu$) and vector ($V_\mu$) currents but not of pseudoscalar ($P$) and scalar ($S$) densities.

- $Z_m$, $Z_P$, $Z_S$, have anomalous dimensions and their calculation introduces a dependence on renormalization scheme and scale.

- Using the renormalization group running of $m_{\overline{MS}}(\mu)$, we can define a scheme and scale independent quantity

\[
\hat{m} = \lim_{\mu \to \infty} \{ m_{\overline{MS}}(\mu) [2\beta_0 g_{\overline{MS}}^2(\mu)]^{-\gamma_0/2\beta_0} \}
\]

\[
\beta_0 = (11 - 2n_f/3)/16\pi^2
\]

\[
\gamma_0 = -8/16\pi^2
\]
LATTICE QCD APPROACH

- First principal QCD calculations with 4 input parameters: $\alpha_s \leftrightarrow a, m_u, m_d, m_s$

- Define lattice quark masses using the AWI and VWI

- Calculate the masses $M_H$ of mesons and baryons as a function of $a, m_i$. Fit $M_H$ versus $m_i$ using forms predicted by $\chi PT$ and extrapolate (interpolate) to physical hadrons to get unrenormalized $m_u, m_d, m_s$.

- Calculate (non-perturbatively) $Z_m, Z_A, Z_P$

- Determine $m_R = Z \times m$

- These masses, $M_H$ and $m_R$, have corrections of $O(a^{n+1})$ if the lattice action and operators are improved to $O(a^n)$. Extrapolate to $a = 0$

- Limitation: Neglect electromagnetic effects $\Rightarrow$ work in isospin symmetric limit $m_u = m_d$ and
CHIRAL EXPANSION

Physical $m_u$ and $m_d$ require very large (128$^4$) lattices and computer time. Currently, resort to a chiral expansion

\[ aM_\Delta(a, m_i, m_j, m_k) = A_\Delta(a) + B_\Delta(a) (m_i + m_j + m_k)_R + C_\Delta(a) (m_i + m_j + m_k)_R^2 + \ldots \]

\[ aM_\rho(a, m_i, m_j) = A_\rho(a) + B_\rho(a) (m_i + m_j)_R + C_\rho(a) (m_i + m_j)_R^2 + \ldots \]  

1

- Physical quark masses $(m_q)_R$ are determined from $A_H(a), B_H(a), C_H(a), \ldots$ by extrapolating (interpolating) Eq. 1 to physical values of $M_H$.

- Different $M_H$ should give the same $m_u, m_d, m_s$ up to corrections of $O(a^{n+1})$

- Extrapolate either $A_H(a), B_H(a), C_H(a), \ldots$, OR $m_q(a)$ to $a = 0$ to remove discretization errors

- Self-consistent determination of quark masses is equivalent to validation of the hadron spectrum.
SYSTEMATIC ERRORS

- Finite Volume: $M_\pi L \geq 5$ for quenched theory
- Chiral fits and extrapolation to $(m_u + m_d)/2$
- $M_K$ with $m_1 \approx m_2 \approx m_s/2$
- Continuum extrapolation to remove discretization errors
- Non-perturbative versus perturbative determination of $Z_m, Z_P, Z_A$

- Quenched approximation
  
  Hadrons do not take on physical masses
  
  Hadrons are stable – no width
  
  Quenched chiral logs; $\eta' \ (\implies m_i > m_s/2)$
RENORMALIZATION CONSTANT

Perturbative versus Non-perturbative

- The variation in the continuum limit is $\approx 2 - 5\%$.
CHIRAL EXTRAPOLATION

UKQCD-ALPHA (hep-lat/9906013; NPB571 (2000) 237)

Fig.a) \[ R = f_K/\langle 0|P|\pi\rangle \text{ and } (m_s + \bar{m}) = R M_K^2 \]

Fig.b) \[ (m_s + \bar{m}) \text{ in units of constant } \ast f_K \]
CONTINUUM EXTRAPOLATION

\[ \frac{m_u + m_d}{2} \text{ in } \overline{MS} \text{ scheme at } \mu = 2 \text{ GeV} \]

- Figure from CP-PACS (hep-lat/0004010 revised).
- Expected errors \( O(g^2a) \). Fit used is \( O(a) \)
- QUENCHED: \( 4.36^{+0.14}_{-0.17} \) MeV
- \( n_f = 2 \): \( 3.44^{+0.14}_{-0.22} \) MeV
CONTINUUM EXTRAPOLATION

$m_s(M_K)$ in $\overline{MS}$ scheme at $\mu = 2$ GeV

- Figure from CP-PACS (hep-lat/0004010 revised).
- Expected errors $O(g^2a)$. Fit used is $O(a)$
- QUENCHED: $110^{+3}_{-4}$ MeV
- $n_f = 2$: $88^{+4}_{-6}$ MeV
CONTINUUM EXTRAPOLATION

\[ m_s(M_\phi) \text{ in } \overline{MS} \text{ scheme at } \mu = 2 \text{ GeV} \]

- Figure from CP-PACS (hep-lat/0004010 revised).
- Expected errors \( O(g^2a) \). Fit used is \( O(a) \)
- QUENCHED: \( 132^{+4}_{-6} \) MeV
- \( n_f = 2 \): \( 90^{+5}_{-11} \) MeV
RECENT QUENCHED RESULTS

$m_q$ in $\overline{MS}$ scheme at $\mu = 2$ GeV

<table>
<thead>
<tr>
<th>Action</th>
<th>$\bar{m}$</th>
<th>$m_s(M_K)$</th>
<th>$m_s(M_\phi)$</th>
<th>scale $1/a$</th>
</tr>
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<tbody>
<tr>
<td>Summary 1997 [1]</td>
<td>3.8(1)(3)</td>
<td>99(3)(8)</td>
<td>111(7)(20)</td>
<td>$M_\rho$</td>
</tr>
<tr>
<td>APE 1999 [3]</td>
<td>O(a) SW</td>
<td>4.8(5)</td>
<td>111(9)</td>
<td>$M_K$</td>
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<td>JLQCD 1999 [5]</td>
<td>Staggered</td>
<td>4.23(29)</td>
<td>106(7)</td>
<td>$M_\rho$</td>
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<td>ALPHA-UKQCD 1999 [6]</td>
<td>O(a) SW</td>
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<td>97(4)</td>
<td>$f_K$</td>
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<td>RIKEN-BNL 1999 [7]</td>
<td>DWF</td>
<td></td>
<td>95(26)</td>
<td>$f_\pi$</td>
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<tr>
<td>QCDSF 1999 [8]</td>
<td>O(a) SW</td>
<td>4.4(2)</td>
<td>105(4)</td>
<td>$r_0 \approx M_\rho$</td>
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<tr>
<td>QCDSF 1999 [8]</td>
<td>Wilson</td>
<td>3.8(6)</td>
<td>87(15)</td>
<td>$r_0 \approx M_\rho$</td>
</tr>
<tr>
<td>CP-PACS 2000 [9]</td>
<td>Iwasaki+SW</td>
<td>4.4(2)</td>
<td>110(4)</td>
<td>$M_\rho$</td>
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<td>$n_f = 2$</td>
<td>3.44$^{+1.14}_{-0.22}$</td>
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Table 1: SW stands for the Sheikhholeslami-Wohlert action.

- $\sim 10\%$ variation depending on quantity used to fix $1/a$
- $\sim 20\%$ difference $m_s(M_{K_1})$ versus $m_s(M_\phi) \approx m_s(M_{K*})$
- $r_0$ defined by $r^2 \partial V(r)/\partial r |_{r = r_0} = 1.65$
SUMMARY OF LATTICE RESULTS

$m_q$ in $\overline{MS}$ scheme at $\mu = 2$ GeV

SYSTEMATIC ERRORS:

- Chiral extrapolation from $\sim m_s$
- Continuum extrapolation: $\sim$ few % error
- Perturbative renormalization constants: $\sim$ few % error

QUENCHED:

- $(m_u + m_d)/2$: 4.2 – 4.8 MeV
- $m_s$: 90 – 143 MeV

$n_f = 2$:

- $(m_u + m_d)/2$: $3.44_{-0.22}^{+0.14}$ MeV
- $m_s$: $89_{-10}^{+6}$ MeV
QCD SUM RULES

Consider the 2-point correlator for an operator $J$:

$$\Psi(q^2) = i \int d^4 x e^{i q \cdot x} \langle 0 \mid J(x) J(0) \mid 0 \rangle$$

This function is analytic on the complex $q^2$ plane with a cut along the positive real axis. It is perturbatively calculable for large $|q^2|$ (say $q^2 > s_0$), and away from the cut.

There are two ways to calculate the discontinuity of this correlator across the cut, i.e., $\rho = \text{Im}\Psi/\pi$

- Perturbation theory via OPE (this introduces quark masses in a given renormalization scheme, and a scale).

- By saturating the intermediate state by hadronic states (i.e. evaluating the Hadronic spectral function).

Matching between these two ways cannot be done point by point: an averaging over $q^2$ is required. This can be avoided by (i) calculating the low order moments of $\rho$, integration up to $s_0$ (Finite Energy Sum Rules), or (ii) Borel transform (Borel/Laplace transform sum rule).
“Finite energy sum rule” for $\bar{m}$


Ward Identity for the axial current (isospin projection $\lambda^\pm$)

$$J = \partial^\mu A^\pm_\mu = 2\bar{m} \bar{q} i\gamma_5 \lambda^\pm q$$

$$\Psi_5(q^2) \equiv i \int d^4 x e^{i q \cdot x} \langle 0 | T \{ \partial^\mu A^-_\mu(x), \partial^\nu A^+_\nu(0) \} | 0 \rangle,$$

$$= (m_d + m_u)^2 i \int d^4 x e^{i q \cdot x} \langle 0 | T \{ P^-(x), P^+(0) \} | 0 \rangle$$

- Perturbative QCD: calculated up to $\alpha_s^3$
- Quark and gluon condensates in OPE are small?
- Hadronic spectral function includes $\pi, \pi(1300), \pi(1770)$ resonances, and the $3\pi$ continuum.
- Duality matching of $\int_{s_0}^{s_0} t^n \rho(t) dt$ done at $s_0 \sim 3 \text{ GeV}^2$. 
“Borel Transform sum rule” for $m_s$

(Jamin&Munz, Chetyrkin,Pirjol,& Schilcher, Colangelo...)

Ward Identity for the $\Delta S = 1$ Vector current

$$J = \partial^\mu V_\mu = i(m_s - m_u)su$$

$$\Psi_5(q^2) \equiv i \int d^4xe^{iq\cdot x} \langle 0|T\{\partial^\mu V_\mu^\dagger(x), \partial^\nu V_\nu(0)\}|0\rangle,$$

$$= (m_s - m_u)^2 i \int d^4xe^{iq\cdot x} \langle 0|T\{S^\dagger(x), S(0)\}|0\rangle$$

- Perturbative QCD: calculated up to $\alpha_s^3$
- Quark and gluon condensates in OPE are small?
- Hadronic spectral function includes $K^*_0(1430), K^*_0(1950)$ resonances, and the $K\pi$ continuum.
- Duality matching of $\int_0^\infty e^{-t/u} \rho(t)dt$ at $s_0 \sim 5.5$ GeV$^2$.

MASTER EQUATION ($u^3\hat{\Psi}''(u)$ is the twice subtracted correlator of scalar currents calculated perturbatively)

$$u^3\hat{\Psi}''(u) = \int_0^{s_0} e^{-s/u} \rho_{hadronic} ds + \int_{s_0}^\infty \rho_{PQCD} ds$$
UNCERTAINTIES IN THESE SUM RULES

- Convergence of Perturbative QCD expressions.
- OPE: Contribution of condensates
- Choice of (Stability with respect to) the matching scale $s_0$. A large $s_0 \Rightarrow$ improved PQCD. On the other hand it has to be within the range of experimental information on hadronic spectral function.
- Ansatz for the hadronic spectral function.
- Overall normalization of the hadronic spectral function.
SUMMARY: SUM RULES

SR results discussed by N. Paver and K. Maltman

- \( m_u + m_d \) from pseudoscalar FESR (PT to \( O(\alpha_s^3) \))
  - Bijnens, Prades, Rafael: \( m_u + m_d = 8.7 \pm 1.8 \text{ MeV} \)

- \( m_s \) from Scalar SR (PT to \( O(\alpha_s^3) \))
  - Jamin 98: \( m_s = 116 \pm 22 \text{ MeV} \)
  - CFNP 98: \( m_s = 104 \pm 12 \text{ MeV} \)
  - Maltman 99: \( m_s = 115 \pm 8 \text{ MeV} \)

- \( m_s \) from \( \tau \) decay SR (PT: \( O(\alpha_s^2) \) + guess for \( O(\alpha_s^3) \))
  - Pich & Prades 99: \( m_s = 114 \pm 23 \text{ MeV} \)
  - Kambor & Maltman 00: \( m_s = 115 \pm 20 \text{ MeV} \)

Limitations for Scalar and PS sumrules:

- slowly converging PT even at \( Q = 2 \text{ GeV} \)
- Insufficient information on hadronic spectral function.

Future:
\( \tau \) decay SR: More data for \( \rho \) in the range \( Q = 1 - 1.4 \text{ GeV} \)
**SUMMARY (\(\overline{MS}; \mu = 2 \text{ GeV}\))**

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Proposed estimates (DPF2000)

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