Oval Expansions of Supersymmetric
Gauge Theories via String Theory
and S-duality

U-duality

IIB string ← S-duality → N=4 super YM

\begin{array}{c}
\text{AdS/QFT} \\
\text{maximally extended supergravity} \\
\text{(quantum gravity)}
\end{array}

\begin{array}{c}
\text{OPE (finite \( \alpha_s \))} \\
\text{S-duality}
\end{array}

\begin{array}{c}
correlators, automorphic functions + scattering \\
\text{space-time structure > unitarity properties} \\
\text{non-renormalization thms}
\end{array}
Holographic Scattering

string state $\leftrightarrow$ composite operator

String (unamputated)

\[ g_s = \frac{1}{4\alpha} \quad g_s^2 \]

\[ \frac{g^4}{\alpha'}^2 = \lambda \]

\[ \alpha' \to 0 \quad g_s \to 0 \quad N \to \infty \quad \lambda = g^2 N \to \infty \]

(planar, strong coupling)

- we shall explore alternate limit

$E, \lambda \to 0$ $E \to$

- to probe finite $\lambda, N$ we must integrate out at higher energies in bulk and decrease string scale + vice-versa
Holographic Scattering

\[ \hat{\prod}_{j=1} \left( \frac{\delta}{\delta \phi_{0,j}}(\bar{z}_j) \right) e^{i S_{\text{string}}[\phi(\phi_0)\phi_0]} + i S_{\text{boundary}}[\phi_0] \]

\[ = \langle \hat{\prod} \mathcal{O}(\bar{z}_j) \rangle \]

\( \phi_0 \) (string field) \( \mapsto \mathcal{O} \) (composite operator)

\( \rightarrow \) 'strong form' \( S_{\text{string}} \rightarrow S_{\text{quantum}}^{\Pi B} \)

\( N^2 \rightarrow 0 \) \( N^2 \) finite

\[ \text{resummed} \quad \rightarrow \quad \text{expanded} \]

\[ + \]

\[ + \]

\[ + \]

\[ + \]

\[ + \]

\[ + \]
\[ N = 4 \leftrightarrow \text{II}B \text{ Matching} \]

\[ \begin{array}{c}
\quad = \quad <\text{Tr} \, q^2(x) \, \text{Tr} \, q^2(y)> \\
\quad = \quad \langle \overset{\bar{x}}{x} \overset{\bar{y}}{y} \rangle \\
\quad \quad \quad \text{propagating sugra fields}
\end{array} \]

\[ \begin{array}{c}
\quad = \quad \langle \overset{\bar{y}}{Y} \overset{\bar{z}}{Z} \rangle \\
\quad \quad \quad \text{boundary space}
\end{array} \]

\[ \text{\Rightarrow \ 2-, 3-Point Correlators are free field ones not a 'strong' check of duality} \]

\[ \text{\Rightarrow \ 4\leq n\text{-point interacting (in QFT may be given a spectral representation )} } \]

\[ \text{\Rightarrow \ quantum \ corrected \ S\text{-matrix does not produce corrections to } 2, 3\text{-pt.}} \]
\[ S_{N=4} + S \int i A^2 + S \nu_{0,1} \partial^0 \]

Finiteness of \( N=4 \) → divergent

Boundary ctms: \( S \phi_0 \partial^k \phi_0 \)

Field theory:
\[ \langle Tr \phi^k(\xi) Tr \phi^l(-\xi) \rangle = T^{\partial} \phi \]
\[ S_{\text{eff}} \frac{1}{\nu^2 (e-k)^2} \]

String theory:
\[ \phi_0 \rightarrow \phi_0 = \int d^4 \xi \int d^\nu z \Delta \Delta g^{\mu \nu} \]
\[ = \frac{c}{12 \nu - 2} \Delta \quad \text{\( \nu \) subtract distributionally} \]

Cutoff: \( \xi_0 \geq \xi \)
\[ d\xi^2 = \frac{c}{\xi_0} \left( d\nu^2 + d\xi^2 \right) \]

UV

IR
Convergence Properties

\[ \prod_{i} \gamma_{i}(x_{i}) = N^{2} F(\lambda, x_{i}) + O(1) \]

- Unitarity positivity conditions from string theory: evidence for cuts
- Requires quantum superstring theory
- \( \mathbb{Z}_{2} \) ambiguity: \( \alpha'^{2} \leftrightarrow \lambda \)
  \( \lambda' \rightarrow e^{2\pi i} \lambda \)
String Expansion (covariantized)

\[ S = \int d^6 x \sqrt{g} \left\{ \frac{1}{2} \frac{1}{\beta^2} R^4 + \sum_{k=0}^8 \epsilon^2 \left( \alpha' R^k + \kappa \tilde{F} \right) \right\} \]

- hard to compute w/ path integral

form:

- eight derivatives

\[ r = \frac{\theta}{2\pi} + \epsilon^{-\Phi} \]

string coupling

\[ \overset{\text{inequivalent vacua}}{\sim} \]

- find form via \( r \rightarrow \frac{\alpha t + b}{c r + d} \) invariance and further physical constraints?

\( \Rightarrow \) perturbative truncation property is related through AdS to \( \lambda_{m,n} \) properties
Vertices

\[ \lambda = g_\text{im}^2 N = \frac{R_k}{\alpha'} \quad g_\text{em} = \frac{1}{4\pi} g_\text{im}^2 \]

\[ \text{from} \quad \int d^4 x \, \mathcal{L} \quad \alpha'(Q)^k \quad R^a_k \quad \Gamma_k \quad \gamma \]

\[ \text{local} \quad \text{in} \quad \text{bulk} \]

\[ V_q \times \Delta_{\mu \nu} \times \Gamma \]

\[ \nabla \]  

\[ \text{OPE into } g_0 \]

\[ \Delta \]

\[ \text{Bulk-Bdy kernel} \]
Space-time Structure

\[ \langle \prod_{j=1}^{n} T_{\mu_{j}}^{\nu_{j}}(x) \rangle = \sum_{k=0}^{\infty} \Omega^k \]

built out of the integrals

\[ I_{\alpha} = \sum \frac{\text{d}^d z_0}{2 \pi} - \prod_{x_{\alpha}} \left( z_0 \frac{z_0}{(z - x_{\alpha}^2)^2} \right) \]

- includes massless exchange
- box diagrams in different dimensions
  (by conformal invar. \( \rightarrow \) triangle diagrams)

- all calculated

- via AdS/CFT these are dual to Feynman diagrams
- $n > 4$-point

- generic in holographic string theory

- MUCH easier than Feynman diagrams

- contains unitarity information

- matches with k-space

- particle interpretation

- orthonormal basis? yes in CFTs
\( R^4 \) Non-Renormalization

\[
S = \int_0^\infty f_G R^4 \left[ a \tau_2^{3/2} + b \tau_2^{-1/2} \right]
\]

NOT fixed by supersymmetry - S-duality in 5 string

SUGRA at \( L = 2 \) (simple end result)

\[
R^4 \left\{ s^2 + \text{perms} \right\} \phi^3 \Rightarrow \text{proves } R^4 \text{ non-renorm}
\]
Limits

I. \( \lambda = g^2 N_c \), \( N_c \to \infty \) \( \sim \lambda \ll N_c \)

\[ \uparrow \]

classical gauged supergravity

II. \( \lambda = g^2 N_c \), \( N_c \) finite

\[ \downarrow \]
IIB superstring theory

III. \( \lambda = 0 \) (\( N_c \) finite) \( \Leftrightarrow \) FT exists here

\[ \downarrow \]
IIB on point compactification (\( R^4 \times \mathbb{C} \))
IIB in tensionless limit (\( \alpha' \to \infty \))

IV. \( \frac{\lambda}{N_c^2} = \frac{1}{\alpha'^2} \to 0 \) \( \Rightarrow N_c \to \infty \) \( \sim \lambda \ll N_c^2 \)

\[ \downarrow \]
dyonic coupling

\[ \downarrow \]
topological holography \( \Leftrightarrow \) IIB
(\( \sim N = 2(4) \) string)
\( N = 4 \quad g_{ym}^2 \rightarrow \frac{1}{g_{ym}^2} \)

planar expansion

\[
\left< \prod_{j=1}^{N} \sigma(\tilde{x}_j) \right> = \sum_{g=2} N^{2(1-g)} F_g(\gamma^i \tilde{x}_i) + \text{instantons}
\]

parameters \( N \), \( \lambda = g_{ym}^2 N \)

S-duality is: \( \lambda \rightarrow \frac{N^2}{\lambda} \) \quad \( N \rightarrow N \) \quad \text{generally}

\( g_{ym}^2 \rightarrow \frac{1}{g_{ym}^2} \) \quad \( N \rightarrow N \)

→ S-duality does not commute with large \( N \)

Question: What is S-dual limit of the large \( N \) expansion \((N \rightarrow \infty)\)?
Graphically:

$S$-dual

$\Rightarrow$

planar

non-planar

+ 'instantons'

re-expand on dyon degrees of freedom

string description $\rightarrow N=4$ topological
Genus Expansion of $N=4$

**SUGRA:**

$$N^2 a(\tilde{x}_j)$$

Genus 0 (massive modes)

$$\frac{1}{2} \left( \frac{2}{N^2} \right)^{3/2} \left[ a_0^{(0)}(\tilde{x}_j) + \frac{1}{\lambda} a_0^{(2)}(\tilde{x}_j) + \ldots \right]$$

Genus 1

$$N \left( \frac{2}{N^2} \right)^{1/2} \left[ a_1^{(0)}(\tilde{x}_j) + \frac{1}{\lambda^{3/2}} a_1^{(3)}(\tilde{x}_j) + \ldots \right]$$

'top' genus in $\mathbb{R}^4$

- demand existence of expansion for

$$\lambda_0 = \left( \frac{2}{N^2} \right)^{-1} = \text{fixed, } N \text{ large}$$

and non-renormalization thms
structure

\[
\begin{array}{cccc}
 g = 0 & g = 1 & g = 2 & g = 3 \\
 R^4 & x & x & \\
 q R^4 & & & \\
 q^2 R^4 & x & x & \\
 q^3 R^4 & x & x & \\
 q^4 R^4 & x & \\
\end{array}
\]

- \( g_{\text{max}} = \frac{1}{2} (k+2) \) \( k \) even
- \( \frac{1}{2} (k+1) \) \( k \) odd

related to large \( N_c \) via 4d S/CFT of \( N = 4 \)

- terms \( \omega \), \( g > g_{\text{max}} \), diverge individually in dual limit

- analog of non-renorm at \( \lambda \to 0 \)

\[ <W(\kappa_i)> = 1 > \lambda^{-3/2} \lambda^{-2} \text{ etc.} \]
Supergravity Properties

\[ A_{\text{4}}^{[n=8]} (k_i, \xi_i) = R^4 \left\{ \begin{array}{c}
2 \quad 3 \\
1 \quad 4 \\
\end{array} \right. + \text{perms} \\
\end{array} \\
+ s^2 + \text{perms} \\
\begin{array}{c}
+ \text{perms} \\
s^4 + \text{perms} \\
\end{array} \right. \quad \text{for any } d \text{ (helicity techniques in } d=4 \text{)}

\text{[after adding diagrams]}

\approx \epsilon_{L} \epsilon_{L} (k \cdot L) \nonumber \epsilon_{(k \cdot L)}

+ \ldots + s^2 + \ldots + \ldots (3 \text{ loops }) \left\{ \\
\right.

\text{• } \phi^3 \text{ structure at } L \leq 2

\text{• Ladder diagrams have explicit } 4(l+L) \text{ powers of momenta extracted}
\text{ (finite in } d \leq 6 \text{)}

\text{• potential } \Omega^3 R^4 \text{ at } L = 3 \text{ naively}
\text{ (but complete amplitude not known)}
\text{ [in conflict with duality if mfo doesn't decouple]}
Quantum $N=8$ S-gravity

- S-duality and dual planar via AdS/QFT indicate cancellations in perturbation theory
  finite in $d \leq 6$

features:

1) found in string-inspired regulator (preserving S-duality remnant)
2) effective tensor property $\lambda^4(1+c)$
3) dimension independent $\lambda$ regulator free?
4) resummation question
5) superconformal ghost insertions related to cancellations in limit?
Conclusions

- finite coupling \( (\beta, N_c) \) effects in \( N=4 \) SYM through AdS duality
- tests of correspondence
  - non-renorm thems, space-time structure unitarity
- holographic renormalization of UV
- mechanism(s) for extended finiteness properties in maximal sugra
  - field theory check requires leading divergence calc in \( d=10 \) (+ L=3?)
  - applications to M-theory in d=11 corner
- IIB graviton scattering related to \( N=4 \)
  - via 'strong' conjecture and instantons related to perturbative via S-duality
• dual planar expansion in $N=4$
  • new regime in AdS duality
  • convergence implied by S-duality
  • $N=4$ topological string

• non-renormalization thms
  • via sugra/string
  • from strong coupling

• space-time structure (generic) of correlations
  • dual to Feynman diagrams

Future:
- free-field limit $\lambda \to 0$ through string?
- applications to geometry $\bowtie K3$
- less supersymmetry
A primary message:

IIB superstring S-matrix can be obtained without a direct path integral calc.