Help for problem 1.4

To review: Let $P(n)$ be the probability distribution for discrete variable, n.

The 1^{st} moment is $\langle n \rangle = \sum_n n\, P(n)$

The r^{th} moment is $\langle n^r \rangle = \sum_n n^r P(n)$

The 1^{st} moment is the mean. The 2^{nd} moment helps evaluate the RMS or standard deviation: $\sigma = \langle (n-\langle n \rangle)^2 \rangle = \langle n^2 \rangle - \langle n \rangle^2$.

The 3^{rd} moment can be used to get the "skew" $= \langle (n-\langle n \rangle)^3 \rangle / \sigma^3$ which measures if the distribution leans in one direction about the mean.

Instead of evaluating the moments directly, it can be useful to use a "moment generating function." That's the approach taken here. The factorial moment generating function is one kind that is useful for discrete variables:

$$F(a) \equiv \langle (1+a)^n \rangle$$

a is a real auxiliary variable of no particular physical significance.

$$= \sum_n (1+a)^n P(n)$$

use binomial expansion of $(1+a)^n$:

$$= \sum_n \sum_{r=0}^{n} \frac{n(n-1)(n-2)\ldots(n-r+1)}{r!} a^r P(n)$$

The numerator is called the "r^{th} factorial moment" which we'll write as $n^{(r)}$ (not to be confused with n^r):

$$F(a) = \sum_n \sum_{r=0}^{n} n^{(r)} \frac{a^r}{r!} P(n)$$

Here are some of the terms. Note: I'm only substituting values for n in the $P(n)$ factor so you can see the pattern.

$$F(a) = P(0) +$$

$$P(1) + n a P(1) +$$

$$P(2) + n a P(2) + \frac{n(n-1)}{2!} a^2 P(2) +$$

$$P(3) + n a P(3) + \frac{n(n-1)}{2!} a^2 P(3) + \frac{n(n-1)(n-2)}{3!} a^3 P(3)$$

and so on...

Add the columns:

$$= 1 \qquad = \langle n \rangle a \qquad = \langle n(n-1) \rangle \frac{a^2}{2!} \qquad = \langle n(n-1)(n-2) \rangle \frac{a^3}{3!}$$

(To get this, I used $n P(n) = 0$ for $n=0$ + $n(n-1)P(n) = 0$ for $n=0,1$ etc)

So,
$$F(a) = \sum_{r=0}^{\infty} \langle n^{(r)} \rangle \frac{a^r}{r!}$$

The Taylor series expansion of $F(a)$ about $a=0$ is:

$$F(a) = \sum_{r=0}^{\infty} \frac{d^r F}{d a^r} \frac{a^r}{r!} \quad \text{so}$$

$$\langle n^{(r)} \rangle = \frac{d^r F}{d a^r} \quad \checkmark$$

as you can see, F is well named.

1.4 For our problem $P(n) = (1-u)u^n$

$$F(a) \equiv \sum_{n=0}^{\infty} (1+a)^n (1-u) u^n$$

(a) Show that $F(a) = \dfrac{1}{1 - \frac{ua}{1-u}}$

Assume that a has a value such that the series can be conveniently summed.

(b) Show that $\langle n^{(m)} \rangle = m! \langle n \rangle^m$