Filling in the blanks in deriving an expression for the spherical harmonics

Recall that we defined the simultaneous eigenkets of $L^2$ and $L_z$: $L_z |\ell m\rangle = m \hbar |\ell m\rangle$ and $L^2 |\ell m\rangle = \ell (\ell + 1) \hbar^2 |\ell m\rangle$ where $\ell$ is a non-negative integer and for a given $\ell$, $m$ assumes $2\ell + 1$ integer values ranging from $-\ell$ to $\ell$. The coordinate space representation in spherical coordinates yields the spherical harmonics denoted by $Y^m_\ell$: $\langle \theta \phi |\ell m\rangle \equiv Y^m_\ell$.

Recall that $L_z = -i\hbar \partial / \partial \phi$ and therefore,

$$\langle \theta \phi | L_z |\ell m\rangle = -i\hbar \frac{\partial}{\partial \phi} \langle \theta \phi |\ell m\rangle = -i\hbar \frac{\partial Y^m_\ell}{\partial \phi} = m \hbar Y^m_\ell.$$

Answer to Jai Salzwedel’s question, “how do we know this?”: The way to justify the above is to start from the postulate (p116)

$$\langle x | P_x | x' \rangle = -i \hbar \frac{\partial}{\partial x} \delta(x - x').$$

Note that this allows one to establish that $\langle x | P_x | \psi \rangle = -i \hbar \frac{\partial}{\partial x} \psi(x)$. Please do this without looking at the book. It is clear that $\langle \vec{r} | L_z | \vec{r}' \rangle = \langle \vec{r} | XP_y - Y P_x | \vec{r}' \rangle = -i \hbar \left( x \frac{\partial}{\partial y} - y \frac{\partial}{\partial x} \right) \delta(\vec{r} - \vec{r}').$

Now interpret this equation in spherical polar coordinates and write this as $-i \hbar \frac{\partial}{\partial \phi} \delta(\vec{r} - \vec{r}')$.

We can denote $|\vec{r}\rangle$ the eigenfunction of $\vec{R}$ by $|\vec{r}\theta\phi\rangle$. Now by the same manipulations as in the case of $P_x$ (using the completeness of $|\vec{r}\rangle$) we can show that

$$\langle r, \theta, \phi | L_z | \psi \rangle = -i \hbar \frac{\partial}{\partial \phi} \psi(r, \theta, \phi).$$

We can drop $r$ since we are dealing with the angular parts only and use $|\theta, \phi\rangle$. Replacing $|\psi\rangle$ by $|\ell, m\rangle$ yields the required result.

The spherical harmonics are obtained using the lowering operator. One first shows as done in the lecture and on page 334 that $Y^\ell_\ell \propto e^{i \ell \phi} \sin^\ell \theta$ and then applies the lowering operator

$$L_- = -\hbar e^{-i \phi} \left( \frac{\partial}{\partial \theta} - i \cot \theta \frac{\partial}{\partial \phi} \right)$$

repeatedly. Suppressing the factor of $\hbar$ we have

$$L_- \left( e^{i m \phi} f(\theta) \right) = -e^{-i \phi} \left( \frac{\partial}{\partial \theta} - i \cot \theta \frac{\partial}{\partial \phi} \right) f(\theta) e^{i m \phi} = -e^{-i(m-1)\phi} \left( \frac{d}{d \theta} + m \cot \theta \right) f(\theta).$$

It is useful to note the following equality:

$$\frac{d}{d \theta} \left( f(\theta) \sin^m \theta \right) = \sin^m \theta \frac{df}{d \theta} + m \sin^{m-1} \theta \cos \theta f = \sin^m \theta \left( \frac{d}{d \theta} + m \cot \theta \right) f(\theta).$$
Therefore, we can write using the above equality and \( d/d\theta = -\sin \theta \frac{d}{d\cos \theta} \)

\[
\left( \frac{d}{d\theta} + m \cot \theta \right) f(\theta) = -\sin^{-m+1} \theta \frac{d}{d\cos \theta} (f(\theta) \sin^m \theta) .
\]

We can therefore write

\[
L_- (e^{i\phi} f(\theta)) = e^{i(m-1)\phi} \frac{g(\theta)}{\sin^{-m+1} \theta} \frac{d}{d\cos \theta} (f(\theta) \sin^m \theta) .
\]

Since this is of the form \( e^{i(m-1)\phi} g(\theta) \) we can apply \( L_- \) again with \( m \to (m - 1) \). Thus we obtain (exercising alertness about what you are doing)

\[
L_-^2 (e^{i\phi} f(\theta)) = L_- (e^{i(m-1)\phi} g(\theta))
\]

\[
= e^{i(m-2)\phi} \cos^{-m+2} \theta \frac{d}{d\cos \theta} (g(\theta) \sin^{m-1} \theta)
\]

\[
= e^{i(m-2)\phi} \cos^{-m+2} \theta \frac{d^2}{d\cos^2 \theta} (f(\theta) \sin^m \theta) . \quad (1)
\]

The lowering operator can be applied \( k \) times and we can obtain

\[
L_-^k (e^{i\phi} f(\theta)) = e^{i(\ell-k)\phi} \cos^{-\ell+k} \theta \frac{d^k}{d\cos^k \theta} (f(\theta) \sin^\ell \theta) .
\]

We can obtain \( Y^\ell_m \) from \( Y^\ell_\ell \propto e^{i\ell\phi} \sin^\ell \) by lowering \( \ell - m \) times:

\[
Y^\ell_m (\theta, \phi) \propto e^{i\phi} \sin^{-m} \theta \left( \frac{d}{d\cos \theta} \right)^{\ell-m} (\sin^\ell \theta) . \quad (2)
\]

The coefficient of proportionality is obtained from normalization:

\[
\langle \ell m | \ell m \rangle = \frac{1}{\pi^2} \int_0^{2\pi} \sin \theta d\theta \int_0^{2\pi} d\phi \langle Y^\ell_m \rangle^* Y^\ell_m = 1 . \quad (3)
\]

With the overall complex phase picked by convention this leads to (12.5.35) on page 335 in the text.