Useful cartesian tensors

In this section the sums run over the three Cartesian directions 1, 2, and 3. We will discuss two commonly used tensors. We will not explicitly verify the transformation properties under orthogonal transformations and check that these are tensors\(^1\). We just focus on getting you to use them effectively.

The Kronecker delta \(\delta_{ij}\) is a second rank tensor. It vanishes for \(i \neq j\) and is unity for \(i = j\). Thus, if \(\hat{e}_i\) are unit vectors along the three directions \(\hat{e}_i \cdot \hat{e}_j = \delta_{ij}\). We also have for any two vectors \(\vec{a}\) and \(\vec{b}\)

\[
\sum_i \sum_j a_i b_j \delta_{ij} = \sum_i a_i b_i = \vec{a} \cdot \vec{b}.
\]

The completely antisymmetric tensor or the Levi-Civita tensor \(\epsilon_{ijk}\) has \(3 \times 3 \times 3 = 27\) components out of which only 6 are nonzero. It is antisymmetric (changes sign) under the exchange of any two indices. Obviously, if any two (or more) indices are the same it vanishes. \(\epsilon_{123}\) is defined to be 1. This implies that \(\epsilon_{231} = \epsilon_{312} = 1;\) these are cyclic permutations of 123. We have \(\epsilon_{213} = \epsilon_{132} = \epsilon_{321} = -1\) since these are obtained by one exchange of two of the indices \(\{123\}\). It is easy to see that

\[
det \begin{pmatrix}
  a_1 & a_2 & a_3 \\
  b_1 & b_2 & b_3 \\
  c_1 & c_2 & c_3
\end{pmatrix}
= \sum_i \sum_j \sum_k \epsilon_{ijk} a_i b_j c_k
\]

The result that the determinant changes sign under the exchange of rows follows from properties of \(\epsilon_{ijk}\).

Quick, what is \(\sum_i \epsilon_{ijk} \delta_{ij}\)? What is \(\sum_i \delta_{ii}\)? What is \(\sum_j \delta_{ij} \delta_{jk}\)?

We can write \(\vec{c} = \vec{a} \times \vec{b}\) as

\[
c_i \equiv (\vec{a} \times \vec{b})_i = \sum_j \sum_k \epsilon_{ijk} a_j b_k
\]

as can be easily verified.

\[
\text{Since } \vec{a} \cdot (\vec{b} \times \vec{c}) = det \begin{pmatrix}
  a_1 & a_2 & a_3 \\
  b_1 & b_2 & b_3 \\
  c_1 & c_2 & c_3
\end{pmatrix}
\]

\(^1\)We are considering tensors in three-dimensional Euclidean space. If one changes the coordinate system by a rotation matrix \(R_{ij}\), a tensor of rank \(r\) denoted by \(T_{i_1i_2...i_r}\) (with \(r\) indices) transforms according to

\[
T'_{i_1i_2...i_r} = \sum_{j_1} \sum_{j_2} \cdots \sum_{j_r} R_{i_1j_1} R_{i_2j_2} \cdots R_{i_rj_r} T_{j_1j_2...j_r}.
\]
we have \( \vec{a} \cdot (\vec{b} \times \vec{c}) = \sum_i \sum_j \sum_k \epsilon_{ijk} a_i b_j c_k \).

An extremely useful identity (that many theorists remember) is
\[
\sum_i \epsilon_{ijk} \epsilon_{i\ell m} = \delta_{j\ell} \delta_{km} - \delta_{jm} \delta_{k\ell}.
\]

This can be verified in many ways. Here is the most pedestrian way. The left-hand side is
\[
\epsilon_{1jk} \epsilon_{1\ell m} + \epsilon_{2jk} \epsilon_{2\ell m} + \epsilon_{3jk} \epsilon_{3\ell m}.
\]
The first term is non-vanishing if \((jk)\) is (23) or (32) and \((\ell m)\) is also (23) or (32). In all these cases the second and third terms vanish. In these four cases the left-hand side has the value ±1. We can substitute the values of \((jk\ell m)\) into the right-hand side and verify the identity. The other possibilities are either the second or the third is non-vanishing and the result follows similarly.

We use this to verify the \(bac - cab\) rule:\footnote{Conversely, if one has proved the vector identity one can use it to verify the given tensor identity.}:
\[
\vec{a} \times (\vec{b} \times \vec{c}) = \vec{b}(\vec{a} \cdot \vec{c}) - \vec{c}(\vec{a} \cdot \vec{b}).
\]
Let \(\vec{b} \times \vec{c} = \vec{d}\). The left-hand side is \(\vec{a} \times \vec{d}\) and its \(k\)th component is \(\sum_j \epsilon_{kji} a_j d_i\). Since \(d_i = \sum_{\ell m} \epsilon_{i\ell m} b_{\ell} c_{m}\) we have
\[
[\vec{a} \times (\vec{b} \times \vec{c})]_k = \sum_{ij} \epsilon_{kji} a_j d_i = \sum_{ij} \sum_{\ell m} \epsilon_{kji} a_j \epsilon_{i\ell m} b_{\ell} c_{m}.
\]
We now have
\[
\sum_i \epsilon_{kji} \epsilon_{i\ell m} = \sum_i \epsilon_{ikj} \epsilon_{i\ell m} = \delta_{k\ell} \delta_{jm} - \delta_{km} \delta_{j\ell}
\]
where the first equality follows from permuting the indices of the epsilon tensor cyclically and the second equality was proved earlier. Thus we obtain
\[
[\vec{a} \times (\vec{b} \times \vec{c})]_k = \sum_{j\ell m} a_j b_{\ell} c_{m} \{\delta_{k\ell} \delta_{jm} - \delta_{km} \delta_{j\ell}\} = b_k (\vec{a} \cdot \vec{c}) - c_k (\vec{a} \cdot \vec{b}).
\]
We have used the result checked earlier that \(\sum_{jm} a_j c_m \delta_{jm} = \vec{a} \cdot \vec{c}\), etc. Since this is true for each \(k\) component the vector identity follows trivially. While I have done this in excruciating detail many of the steps can be skipped even with modest familiarity of these tensors.

Many results can be derived using these and similar identities and are worth learning to ease vector manipulations and identities. For example, use this identity to determine
\[
(\vec{a} \times \vec{b}) \cdot (\vec{c} \times \vec{d}).
\]
Vector analysis identities useful in electromagnetic theory can be derived if we note that

\[ [\vec{\nabla} \times \vec{A}]_i = \sum_{jk} \epsilon_{ijk} \partial_j A_k \]

where \( \partial_j \equiv \frac{\partial}{\partial x_j} \). It is often convenient to use the Einstein summation convention and eliminate the ugly \( \sum \) symbols.

What is \( \vec{\nabla} \times \vec{\nabla} f \)? If we let \( \vec{A} = \vec{\nabla} f \) (as you get familiar with these manipulations you can discard such crutches)

\[ [\vec{\nabla} \times \vec{A}]_i = \epsilon_{ijk} \partial_j A_k = \epsilon_{ijk} \partial_j \partial_k f . \]

The second equality follows since \( A_k = \partial_k f \). The sum over \( j \) and \( k \) obviously vanishes since the two partial derivatives are symmetric under the exchange of \( j \) and \( k \) while the epsilon tensor is antisymmetric. You should check this once clearly and remember it. Thus the curl of a gradient vanishes.

By considering the \( k^{th} \) component and using the completely antisymmetric tensor verify the tough-looking identity

\[ \vec{\nabla} \times [\vec{a} \times \vec{b}] = (\vec{b} \cdot \vec{\nabla}) \vec{a} - (\vec{a} \cdot \vec{\nabla}) \vec{b} + \vec{a}(\vec{\nabla} \cdot \vec{b}) - \vec{b}(\vec{\nabla} \cdot \vec{a}) . \]