Some numbers to be estimated

(a) Find the electric field at the proton due to the electron or vice versa in V/m. You can do the calculation classically with numbers from experiments/quantum calculation. (1 point)

\[ \frac{e}{(4\pi \epsilon_0 a_0^2)} \] yields \(5.1 \times 10^{11} \text{ V/m} \).

(b) Consider a CW laser with an intensity \(5 \times 10^{14} \text{ W/cm}^2\). Find the mean electric field at this intensity and compare it with your result in (a). (2 points)

From an elementary electromagnetic theory textbook we have for the average Poynting vector (energy flux density) the expression \(\frac{1}{2} \epsilon_0 c E^2\) and substitution yields \(6.1 \times 10^{10} \text{ V/m} \).

(c) Find an explicit expression for the magnetic field at the electron in its rest frame due to the moving proton in terms of the radius and angular momentum of the electron. Estimate the field in Tesla for the \(n = 2, \ell = 1\) state of the hydrogen atom. This is useful for thinking about the spin-orbit interaction. (2 points)

In the rest frame of the electron, the proton is in motion, and hence, the electron feels a magnetic field given by \(-\frac{1}{c^2} \vec{v} \times \vec{E}\) where \(\vec{v}\) is the velocity of the electron from Lorentz transforming fields. You can use the Biot-Savart law. Writing the force \(-e\vec{E}\) as the negative gradient of the potential energy \(V\) we have

\[ -e\vec{E} = -r \frac{dV}{dr}. \]

Thus the magnitude of the magnetic field is

\[ \frac{1}{r} \frac{dV}{dr} \frac{1}{emc^2} |\vec{r} \times \vec{p}|. \]

In our case we have

\[ \frac{1}{r} \frac{dV}{dr} = \frac{e^2}{4\pi \epsilon_0 r^3}. \]

This leads to the expression

\[ \frac{1}{mc^2} \frac{e}{4\pi \epsilon_0 r^3} |\ell|. \]

This can be evaluated for the 2p state which has non-vanishing angular momentum and \(\langle r^{-3} \rangle\) can be found. You can get a rough estimate by using \(r = 2a_0\) for the 2p state and write \(r^{-3} \approx \frac{1}{64a_0^3}\). An actual evaluation yields \(\frac{1}{24a_0^3}\). Using \(\hbar\) for \(\ell\) the answer I got was 0.52T. In any event an estimate of about 1T to within a factor of 3 is acceptable.
Other comments: Recall that the magnetic moment due to spin is given by \((-e/m)\vec{s}\) where \(e > 0\) (the minus sign is explicitly included) and \(s\) has units of angular momentum. Also we have used \(g = 2\). Thus the interaction energy \(-\vec{\mu} \cdot \vec{B}\) is given by

\[
H_{s.o.} = -\frac{e}{mc^2} \vec{s} \cdot (\vec{v} \times \vec{E}).
\]

and thus

\[
H_{s.o.} = -\frac{1}{m^2c^2} \frac{1}{r} \frac{dV}{dr} \vec{s} \cdot (m \vec{v} \times \vec{r}) = \frac{1}{m^2c^2} \frac{1}{r} \frac{dV}{dr} \vec{\ell} \cdot \vec{s}.
\]

We now note that the electron in a circular orbit is accelerating and hence the transformation is to an accelerating coordinate system that must be done more carefully. This was done by L. H. Thomas in 1926 (who joined the faculty at Ohio State in 1929) and the phenomenon is called Thomas precession. It reduces the energy by a factor of exactly \(1/2\). You were allowed to ignore this subtle factor.

(d) Estimate the magnetic field in Tesla that must be applied to an electron if the interaction energy with the intrinsic magnetic dipole moment of the electron equals the binding energy in the 1s state. What would the field be if the energy equals \(k_B T\) where \(T\) is room temperature? Compare these to available laboratory fields. (2 points)

It is useful to note that the Bohr magneton is \(5.79 \times 10^{-5} \text{eV/T}\). So if we want \(\mu_B B\) to be 13.6eV the field should be \(13.6/\mu_B\) in Tesla. This yields \(2.4 \times 10^8 T\). At room temperature which is \((1/40)\)eV the field is 430T. Both of these are much larger than static fields that can be obtained in the lab. How is physics different in neutron stars where billions and trillions of Tesla fields can occur?

2. Given \(\psi(\vec{r})\) the (normalized) wave function for a particle in three dimensions what is the probability of finding the particle with the magnitude of its momentum between \(p\) and \(p + dp\)? Evaluate and plot this probability for the ground state of the hydrogen atom. Make sure you specify the units when you plot it. This is not an academic exercise since this has been measured experimentally. (5 points)

We first compute the wave function in momentum space:

\[
\psi(\vec{p}) = \frac{1}{(2\pi\hbar)^{3/2}} \int d^3r \, e^{-i\vec{p} \cdot \vec{r}/\hbar} \psi(\vec{r}).
\]

\(|\psi(\vec{p})|^2 d^3p\) is the probability of finding the momentum of the particle in an infinitesimal volume \(d^3p\) around \(\vec{p}\).
We calculate the momentum-space wave function for the ground state of the hydrogen atom. We know that 
\[ \psi(\vec{r}) = \frac{1}{\sqrt{\pi a^3}} e^{-r/a}. \]
For the integral in coordinate space we can orient the axes any way we choose. Let the z-axis lie along \( \vec{p} \). Clearly, \( \vec{p} \cdot \vec{r} = pr \cos \theta \). This makes the \( \phi \) integral trivial. Performing it we obtain 
\[ \psi(\vec{p}) = \frac{1}{\sqrt{\pi}} \left( \frac{2}{\pi a^3} \right)^{3/2} \frac{1}{\sqrt{(2 \pi a \hbar)^3/2}} \int_0^\infty dr \ r^2 e^{-r/a} \left( \int_0^\pi e^{-ipr \cos \theta / \hbar} \sin \theta d\theta \right). \]
Something you should be able to do in seconds in graduate school (you check this once carefully and remember it) is to change the \( \theta \) integral into 
\[ \int_{-1}^1 d\mu \ e^{-ipr \mu / \hbar} \] by letting \( \mu = \cos \theta \).
The \( \theta \) integral yields 
\[ \frac{\hbar}{ipr} \left( e^{ipr / \hbar} - e^{-ipr / \hbar} \right). \]
We can then do the \( r \) integral in a straightforward way and obtain 
\[ \psi(\vec{p}) = \frac{1}{\pi} \left( \frac{2a}{\hbar} \right)^{3/2} \frac{1}{1 + \frac{a^2 p^2}{\hbar^2}}. \]
The probability for the magnitude of the momentum to lie between between \( p \) and \( p + dp \) is \[ 4\pi p^2 |\psi(\vec{p})|^2. \] This is easily plotted using Mathematica for example.

3. **Standard undergraduate examination problem** A spin-1/2 particle is in the eigenstate of \( S_x \) at time \( t = 0 \) in a field \( \vec{B} = B \hat{z} \). It precesses in this field for a time \( \tau \). At \( t = \tau \) the field is very rapidly (=instantaneously) changed to \( \vec{B} = B \hat{y} \). After another interval of time \( \tau \), \( S_x \) is measured. What is the probability of finding the value \( +\hbar/2 \)? For a 1T field find the time at which the probability to flip is unity for the first time. (5 points)

This is mostly algebra. So organize it so that you can do it efficiently. We denote the gyromagnetic ratio by \( \gamma \) so that the magnetic moment is \( \gamma \vec{S} \). The time evolution for the first period \( \tau \) is controlled by \( -\gamma B S_z = -\frac{\hbar \gamma}{2} \sigma_z \) where \( \omega \equiv \gamma_e B \). The time evolution for the second period is \( H = -\frac{\hbar \omega}{2} \sigma_y \). Since the time-evolution operator is \( e^{-iH\tau / \hbar} \) we have
\[ |\chi(2\tau)\rangle = e^{i\omega \gamma \sigma_y} e^{i\omega \gamma \sigma_z} |\chi(0)\rangle. \]
We use \( e^{i\alpha \hat{n}} = \cos(\alpha) I + i \sin(\alpha) \hat{n} \cdot \vec{\sigma} \) to rewrite the expression above. Let \( C \equiv \cos(\omega \tau / 2) \) and \( S \equiv \sin(\omega \tau / 2) \)
\[ |\chi(2\tau)\rangle = |CI + iS\sigma_y| |CI + iS\sigma_z| |\chi(0)\rangle. \]
\[ |\chi(2\tau)\rangle = |[C^2 I + iCS(\sigma_y + \sigma_z) - iS^2 \sigma_z]| |\chi(0)\rangle. \]
In the second line we have used $\sigma_y\sigma_z = i\sigma_x$. It is easy to write down the matrix now by just adding the four matrices and obtain

$$|\chi(2\tau)\rangle = \begin{pmatrix} C^2 + iCS & CS - iS^2 \\ -CS - iS^2 & C^2 - iCS \end{pmatrix} |\chi(0)\rangle.$$ 

It is easy to check that the matrix is unitary. We can calculate the quantities we were asked to do by doing elementary matrix multiplication. The probability for the spin to be up along $z$ given $|\chi(t = 0)\rangle = |\uparrow\rangle$ is $\cos^4(\omega\tau/2) + \sin^4(\omega\tau/2) = 1 - \frac{1}{2} \sin^2(\omega\tau)$. This means that the probability to be down is $\frac{1}{2} \sin^2(\omega\tau)$. This is never unity. Bad question. I would have asked for the probability to be $1/2$ if I had done the problem before I gave it.

4. You can ignore the spatial degrees of freedom in this problem. Given an electron in a magnetic field $\vec{B}(t) = B_0 \cos(\omega t) \hat{z}$ pointing up along the $X$-axis at time $t = 0$ find the probability of measuring the spin at time $t = \tau$ along the $x$-axis and finding $-\hbar/2$. Does the spin flip for any field? Give a quantitative answer. (5 points)

The magnetic moment is $-\gamma_e \vec{S}$ and so the Hamiltonian is $\frac{\gamma_e \hbar B_0}{2} \cos(\omega t) \sigma_z$. Defining $\omega_0 \equiv \gamma_e B_0$ the Schrödinger equation reads

$$i \begin{pmatrix} \dot{a} \\ \dot{b} \end{pmatrix} = \begin{pmatrix} \frac{\omega_0}{2} \cos \omega t & 0 \\ 0 & -\frac{\omega_0}{2} \cos \omega t \end{pmatrix} \begin{pmatrix} a \\ b \end{pmatrix}.$$ 

These are two decoupled ODEs and are much easier to solve than the case treated in the lecture. We obtain

$$a(t) = a(0) e^{-i \frac{\omega_0}{2} \sin(\omega t)}.$$ 

$$b(t) = b(0) e^{i \frac{\omega_0}{2} \sin(\omega t)}.$$ 

We are given that $a(0) = 1/\sqrt{2}$ and $b(0) = 1/\sqrt{2}$ and therefore

$$|\chi(t)\rangle = \frac{1}{\sqrt{2}} \begin{pmatrix} e^{-i \frac{\omega_0}{2} \sin(\omega t)} \\ e^{i \frac{\omega_0}{2} \sin(\omega t)} \end{pmatrix}.$$ 

We need $|\langle z - |\chi(t)\rangle|^2$ and this is easily evaluated to be

$$\sin^2\left(\frac{\omega_0}{2\omega} \sin(\omega t)\right).$$ 

You should plot this for $\omega > \omega_0$ and $\omega < \omega_0$.

5. Consider a spin-1/2 particle of mass $m$ and charge $q$ in a vector potential $\vec{A}(\vec{r}, t)$. You may recall that the kinetic energy of the Hamiltonian (this is discussed in classical physics in Goldstein) is given by

$$H_K = \frac{1}{2m} \left( \vec{P} - q \vec{A} \right) \cdot \left( \vec{P} - q \vec{A} \right).$$
The interaction of the intrinsic magnetic moment with the magnetic field associated with the vector potential is of the standard form

\[-\frac{q\hbar}{2m} \vec{\sigma} \cdot \vec{B}.

Show that the two terms can be combined neatly into the form

\[H = \frac{1}{2m} \left[ \vec{\sigma} \cdot (\vec{P} - q\vec{A}) \right]^2.

State clearly what gauge you have used and where this assumption entered your calculation. This form can be useful. (5 points)

Start from the second form and use the identity

\[(\vec{\sigma} \cdot \vec{B})(\vec{\sigma} \cdot \vec{C}) = \vec{B} \cdot \vec{C}I + i\vec{\sigma} \cdot (\vec{B} \times \vec{C}).\]

to obtain

\[\frac{1}{2m}(\vec{P} - q\vec{A}) \cdot (\vec{P} - q\vec{A})I + i\vec{\sigma} \cdot [(\vec{P} - q\vec{A}) \times (\vec{P} - q\vec{A})].\]

The first term yields \(H_K\). Using the fact that \(\vec{P} \times \vec{P}\) and \(\vec{A} \times \vec{A}\) vanish (the operators commute) we are left with

\[-\frac{q\hbar}{2m} \vec{\sigma} \cdot \left( \vec{\nabla} \times \vec{A} + \vec{A} \times \vec{\nabla} \right).\]

Since the expression in the brackets is an operator we must evaluate \(\vec{\nabla} \times (\vec{A}f) + \vec{A} \times \vec{\nabla} f\); using the identity (easily established using the antisymmetric tensor) \(\vec{\nabla} \times (\vec{A}f) = f\vec{\nabla} \times \vec{A} - \vec{A} \times \vec{\nabla} f\) we are left with \(f\vec{B}\) and since these are functions of the position we can write it with the order reversed. This establishes the result.

It is important to note that the gauge is not specified and the derivation. If one expands the kinetic term to obtain the \(\vec{L} \cdot \vec{B}\) term we use the \(\vec{\nabla} \cdot \vec{A} = 0\) gauge as is done on page 388 in the text. The term quadratic in \(\vec{A}\) yields diamagnetic effects. This identity is useful in deducing the intrinsic magnetic moment of the electron from the Dirac equation.