Directions: Step through the session and write down answers to the questions/problems in italics as you go. Hand it in at the end of the period for credit. Please work in pairs (but hand in individual sheets) and feel free to check your answers against those from other groups.

I. Using Complex Numbers with MATLAB (20 minutes)

The “MATLAB Cheatsheet V” handout describes the basics of complex numbers in MATLAB (which are used pretty much as you would guess). Do the following by hand first and then check with MATLAB. Recall that if $z = x + iy = re^{i\theta} = r\cos \theta + ir\sin \theta$, then $r = |z| = \sqrt{zz^*} = \sqrt{x^2 + y^2}$ and $\theta = \tan^{-1}(y/x)$.

1. Find the cartesian form $(x + iy)$, the complex conjugate $z^*$, and the real and imaginary parts of $z = 2e^{-i\pi/4}$.

2. Give the polar form (i.e., $re^{i\theta}$) of $1/(1 - \sqrt{2}i)$.

3. [BONUS] The impedance $Z$ of an RLC circuit is given by

$$Z = i\omega L + R + \frac{1}{i\omega C} = |Z|e^{i\phi}.$$ 

Derive expressions for $|Z|$ and $\tan \phi$. Draw a representation of $Z$ in the complex $Z$ plane.

4. [BONUS] Using MATLAB, compare $\cos(x)$ to $\cosh(ix)$ and $\sin(x)$ to $\sinh(ix)$ for several values of $x$ (e.g., $x = \pi/3$). What do you conclude about how $\cos(z)$ is related to $\cosh(z)$ in general? What about $\sin(z)$ and $\sinh(z)$?
II. Convergence of Series in the Complex Plane (25 minutes)

Download complex_plot.m and complex_convergence.m from the 263 webpage to your 263 directory. The first program plots the magnitude (modulus) of a function of the complex number $z = x + iy$. The second program is essentially the same as the test_convergence.m program from Session 4, except now intended for series evaluated with complex $z$ instead of real $x$ (which just means it plots the modulus of the sum).

Take a look at the program listing for complex_plot.m in the editor. What is the function of $z$ to be plotted? Before running the program, sketch what the function would look like as a function of $x$ if $z$ were purely real (i.e., $z = x$ and $y = 0$) and then what it would look like as a function of $y$ if $z$ were purely imaginary (i.e., $z = iy$ and $x = 0$).

Now run the program and look at both the surface plot (figure 1) and the contour plot (figure 2). Where are the “singularities” (the places where the function blows up)? Does this agree with your plots (after taking the absolute value)?

We’ve discussed the convergence of a series expansion of $f(x)$ of the form

$$f(x) = \sum_{n=0}^{\infty} a_n x^n$$

using the ratio test, which gives the radius of convergence $R$ as

$$|x| < R = \lim_{n \to \infty} \left| \frac{a_n}{a_{n+1}} \right| .$$

An alternative way to find the radius of convergence is to consider the function $f(z)$ in the complex $z$ plane, where $z = x + iy$. According to our discussion in class, the “radius of convergence” of an expansion about a given point (e.g., $x = 0$) is given by the radius of the circle centered at that point that just intersects the nearest singularity. What is the prediction for the function here? Does that agree with the ratio test?

Now let’s test these results “experimentally” using complex_convergence.m. What are the first
few terms in the Taylor series of the current function? Modify the test function in the program so that it matches the current function. Run the program and check the convergence for real numbers (e.g., 0.99 and 1.01). What do you conclude about the radius of convergence?

Now try numbers in the complex plane! This is easiest in polar form, e.g., when asked for z, enter something like .99*exp(i*pi/4). What do you find? Predict whether the series will converge for z = 2/3 + 2i/3 and then check by entering this as 2/3 + i*2/3.

[BONUS] Now suppose you change the function to 1/(z^2 - z + 1). Where do you predict the singularities to be? [Hint: Find where the denominator vanishes using the quadratic equation.] Modify the plotting program and verify (or refute!) your answer.

[BONUS] Finally, consider the function 1/(z^4 - z^3 + z^2 - z + 1). How many poles (singularities) do you predict? Modify the plotting program to verify your answer. What is the pattern that you observe among pairs of poles?