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834 Introductory Comments

(0) Prof. Dick Furnstahl, Nuclear Theory Group
- low-energy nuclear physics (as opposed to high-energy relativistic heavy-ion collisions)
- nuclear structure and reactions from microscopic calculations,
- many computational aspects (one reason I'm teaching this course!)

(1) 834 logistics — (use projected web pages)
- 834 home page is communication center for course
  - handouts like Wigner's "Unreasonable..." essay
  - recent changes section — check announcements
- Jump to course description and info
  - texts, prerequisites, material, schedule, grading and office hours
  - * show of hands: How many not available 8:30-10:18am Fridays?
- Assignments, Handouts, Lecture Notes
  - username: physics password: 834
  - must do problems to learn
  - I will try to make homeworks useful, not excessive
    - give feedback!
  - Interact with classmates but don't ask for help too soon
- References
  - several ebooks ⇒ Ebrary: My Bookshelf can be used
  - comment on listed books

(2) Overview of 834 Math Methods
- first time replacement of usual Jackson E+M 1st quarter
- Goals:
  - establish mathematical core compentencies for E+M in particular, but more generally as foundation for all physics
  - learn how and where to look things up (including Mathematica)

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- What topics should 834 cover?
- Consider the first equations in the introduction to Jackson's "Classical Electrodynamics" text: Maxwell's Equations

$$\vec{\nabla}\cdot\vec{D} = \rho$$
$$\vec{\nabla}\times\vec{H} - \frac{\partial \vec{D}}{\partial t} = \vec{J} \qquad \text{all functions of } (\vec{x},t)$$
$$\vec{\nabla}\times\vec{E} + \frac{\partial \vec{B}}{\partial t} = 0 \qquad \underline{\text{local}} \text{ equations (at a point and nearby from derivatives)}$$
$$\vec{\nabla}\cdot\vec{B} = 0 \qquad \Rightarrow \underline{\text{global}} \text{ consequences}$$

To deal with these we need:

- vector calculus (today and PS#1)
  - intrinsically 3-dimensional
  - use different coordinate systems to simplify (geometry)
- differential equations → special functions, eigenfunction expansions (Sturm-Liouville)
  → linear ⇒ superposition → Green's functions
- point sources → delta (generalized) function
- Fourier series and integrals (solving diff. eqs, momentum space, ...)
- complex analysis — contour integrals, dispersion relations
  ⇒ see Wigner essay

- List of topics matches Lea's text well ⇒ use as guide
  - Arfken and Weber as more comprehensive reference

- Many topics and limited time (in and out of class)
  - examples to provide foundation
  - practice looking thing up (eg, special functions) in books, Mathematica, ...

- Issues: • vocabulary — many terms may be new ⇒ ask if unclear
  • notation — become exposed to full variety
  eg, $\hat{x}$ or $\hat{i}$ or $\hat{e}_x$ or ...

* If you know these topics well, consider placing out of 834.

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How is 834 different from a math course?

1. Emphasis on physics intuition rather than mathematical rigor.
   ⇒ derivations and motivations but seldom proofs (in class)

2. Using math as a tool for physics (and E+M in particular)
   ⇒ selective coverage and only as general/abstract as needed (for working "by hand")

3. Mathematica as a tool and substitute, when appropriate

4. Numerical solutions (as opposed to symbolic "analytic")
   - also conceptual simplifications, e.g. operators → matrices
     Hilbert spaces → finite dimensional

- Do sampling of Mathematica capabilities (without explaining)

  ? SphericalBesselJ (and follow help) cf. Abramowich + Stegun
  - plots
  - Series[SphericalBesselJ[3,x], {x,0,5}] small x
    {x, Infinity, 3}] large x (asymptotic)
    // FullSimplify ["O" means "order of"]

  - SphericalBesselJ[2,x]
    // FunctionExpand
  - LegendreP[100,x]

  Notation: brackets matter!
  [≠ (≠ {

  Integrate[Cos[x]/x, {x,-1,2}]
  // TraditionalForm but undefined!
  add PrincipalValue -> True (more soon!)

Vector calculus competencies in short term: define by inside covers of Jackson
- manipulate tensors to derive vector formulas
- div, grad, curl in cylindrical and spherical coordinates
- intuition and application of theorems

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Kronecker Delta function and Levi-Civita (Epsilon) Symbol

• first go through annotated sheet sections 1, 2, 3

• first proof:

$\vec{A}\cdot(\vec{B}\times\vec{C}) = A_i(\vec{B}\times\vec{C})_i = A_i\varepsilon_{ijk}B_jC_k \overset{\varepsilon \text{ can move anywhere}}{=} \varepsilon_{ijk}A_iB_jC_k$

cyclic or even permutation

$= \varepsilon_{kij}A_iB_jC_k = \varepsilon_{kij}C_kA_iB_j = C_k(\vec{A}\times\vec{B})_k = \vec{C}\cdot(\vec{A}\times\vec{B})$

if C_k, A_i, B_j commute

• class vote $\overset{?}{=} \vec{B}\cdot(\vec{C}\times\vec{A})$ ✓ cyclic!

or $\overset{?}{=} \vec{B}\cdot(\vec{A}\times\vec{C})$ ×

• Two quick examples (more from Jackson front cover to try!)

relabel dummy indices $j \leftrightarrow k$; now switch k, j

(1) $\vec{a}\times\vec{a} \Rightarrow (\vec{a}\times\vec{a})_i = \varepsilon_{ijk}a_ja_k = \varepsilon_{ikj}a_ka_j = -\varepsilon_{ijk}a_ka_j$

antisymmetric × symmetric $= 0$

if commuting $= -\varepsilon_{ijk}a_ja_k = 0$

can be interchanged

cf., $[\nabla\times\vec{\nabla}\psi(\vec{x})]_i = \varepsilon_{ijk}\frac{\partial}{\partial x_j}\frac{\partial}{\partial x_k}\psi(\vec{x}) = 0$ but $\frac{\partial}{\partial x_i}x_jf(\vec{x}) \neq x_j\frac{\partial}{\partial x_i}f(\vec{x})$

notation! $(\vec{\nabla})_i = \frac{\partial}{\partial x_i} = \partial_i = \nabla_i$

(2) $[\underbrace{\vec{\nabla}}_{j}\times\underbrace{(\vec{\nabla}\times\vec{a})}_{k}]_i = \varepsilon_{ijk}\frac{\partial}{\partial x_j}(\vec{\nabla}\times\vec{a})_k \overset{?}{=} \cancel{\varepsilon_{ijk}\frac{\partial}{\partial x_j}\varepsilon_{kij}\frac{\partial}{\partial x_i}a_j}$ no! i, j appear more than twice

$= \varepsilon_{ijk}\frac{\partial}{\partial x_j}\varepsilon_{klm}\frac{\partial}{\partial x_l}a_m$ correct

• simplify: eliminate 2 ε's with $\varepsilon_{ijk}\varepsilon_{klm} \overset{\text{permute repeated index to first place}}{=} \varepsilon_{kij}\varepsilon_{klm} = (\delta_{il}\delta_{jm} - \delta_{im}\delta_{jl})$

eliminate δ's

$= (\delta_{il}\delta_{jm} - \delta_{im}\delta_{jl})\frac{\partial}{\partial x_j}\frac{\partial}{\partial x_l}a_m = \frac{\partial}{\partial x_m}\frac{\partial}{\partial x_i}a_m - \frac{\partial}{\partial x_l}\frac{\partial}{\partial x_l}a_i$

commute ∂; identify

$= \frac{\partial}{\partial x_i}\frac{\partial}{\partial x_m}a_m - \frac{\partial}{\partial x_l}\frac{\partial}{\partial x_l}a_i = \frac{\partial}{\partial x_i}\vec{\nabla}\cdot\vec{a} - \nabla^2 a_i$

If time, try more.

$= [\vec{\nabla}(\vec{\nabla}\cdot\vec{a}) - \nabla^2 a]_i$ (we can drop the i's now)

9/2/11 Lecture 1 – treat as if haven't seen it.

(5)

- Project on screen
- follow up in lecture 2 with some matrices, tensors
- How would you implement in Mathematica?

Kronecker Delta Function δ_{ij} and Levi-Civita (Epsilon) Symbol ε_{ijk}

vocabulary (cf. Dirac delta function)

be careful of other ε's (dielectric tensor ε_{ij})

multiple ways to deal with vectors, this one generalizes to tensors of higher rank

1. **Definitions**

$$\delta_{ij} = \begin{cases} 1 & \text{if } i = j \\ 0 & \text{otherwise} \end{cases} \qquad \varepsilon_{ijk} = \begin{cases} +1 & \text{if } \{ijk\} = 123,\ 312, \text{ or } 231 \\ -1 & \text{if } \{ijk\} = 213,\ 321, \text{ or } 132 \\ 0 & \text{all other cases (i.e., any two equal)} \end{cases}$$

second nature

ε_{ijk} or ε_{abc} or ε_{lmn}

- So, for example, $\varepsilon_{112} = \varepsilon_{313} = \varepsilon_{222} = 0$.
- The +1 (or *even*) permutations are related by rotating the numbers around; think of starting with 123 and moving (in your mind) the 3 to the front of the line, to get 312. Do it again with the 2 and you get 231. The −1 (or *odd*) permutations starting with 213 are related to each other the same way; they are related to 123 by interchanging just two of the numbers (e.g., switch the 1 and 3 to get 321).

2. **Applying δ_{ij} and ε_{ijk} to Vectors in Cartesian coordinates** (will not work like this for cylindrical or spherical!)

- Instead of using x, y, and z to label the components of a vector, we use 1, 2, 3. (later, because basis unit vectors are not constant as $\hat{x}, \hat{y}, \hat{z}$ are)
- Then the letters i, j, k, ... can be used as dummy summation variables, running from 1 to 3. (We could use any other letters, like a, b, ...; it is merely a convention.)
- Don't confuse the use of the dummy summation variables i, j, k, each of which can be 1, 2, or 3, with the unit vectors $\hat{\mathbf{i}}$, $\hat{\mathbf{j}}$, $\hat{\mathbf{k}}$. These are two independent notations!
- The dot product of two vectors $\mathbf{A}\cdot\mathbf{B}$ in this notation is

$= A_xB_x + A_yB_y + A_zB_z$

$$\mathbf{A}\cdot\mathbf{B} = A_1B_1 + A_2B_2 + A_3B_3 = \sum_{i=1}^{3} A_iB_i = \sum_{i=1}^{3}\sum_{j=1}^{3} A_iB_j\delta_{ij} = \sum_{ij} A_i \delta_{ij} B_j$$

Be careful if could have more! (numerical applications)

matrix multiplication

Note that there are nine terms in the final sums, but only three of them are non-zero.

$(A)\begin{pmatrix} 1 & & \\ & 1 & \\ & & 1 \end{pmatrix}(B)$

- The i^{th} component of the cross produce of two vectors $\mathbf{A}\times\mathbf{B}$ becomes

$$(\mathbf{A}\times\mathbf{B})_i = \sum_{j=1}^{3}\sum_{k=1}^{3} \varepsilon_{ijk}A_jB_k = \sum_{l=1}^{3}\sum_{m=1}^{3} \varepsilon_{klm} A_l B_m$$

Again, there are nine terms in the sum, but this time only two of them are non-zero. Note also that this expression summarizes three equations, namely for $i = 1, 2, 3$.

3. **Einstein Summation Convention**

- We might notice that the summations in the expressions for $\mathbf{A}\cdot\mathbf{B}$ and $\mathbf{A}\times\mathbf{B}$ are redundant, because they only appear when an index like i or j appears twice on one side of an equation. So we can omit them. Thus

$$\sum_{i=1}^{3}\sum_{j=1}^{3} A_iB_j\delta_{ij} \longrightarrow A_iB_j\delta_{ij} = A_iB_i \quad \text{and} \quad \sum_{j=1}^{3}\sum_{k=1}^{3} \varepsilon_{ijk}A_jB_k \longrightarrow \varepsilon_{ijk}A_jB_k\,.$$

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- Rules: If an index appears (exactly) twice, then it is summed over and appears only on one side of an equation. A single index (called a *free index*) appears once on each side of the equation. So

easy to do if you reuse an index by mistake

$$\textit{Valid: } A_i = A_j\delta_{ij}\,, \quad B_k = \varepsilon_{ikl}A_iC_l \qquad \textit{Invalid: } A_i = B_iC_i\,, \quad A_i = \varepsilon_{ijk}B_iC_j\,.$$

- When you have a Kronecker delta δ_{ij} and one of the indices is repeated (say i), then you simplify it by replacing the other i index on that side of the equation by j and removing the δ_{ij}. For example:

look at δ, find an index, and substitute

$$A_j\delta_{ij} = A_i\,, \qquad B_{ij}C_{jk}\delta_{ik} = B_{kj}C_{jk} = B_{ij}C_{ji}$$

If δ's, use them to eliminate indices

Note that in the second case we had two choices of how to simplify the equation; use either one!

simplification rules (can implement in Mathematica)

1) replace doubled ε's
2) use δ's to eliminate indices

- The triple or box product $\mathbf{A}\cdot(\mathbf{B}\times\mathbf{C})$ can be written

$$\mathbf{A}\cdot(\mathbf{B}\times\mathbf{C}) = \varepsilon_{ijk}A_iB_jC_k = \varepsilon_{kij}A_iB_jC_k = \varepsilon_{kij}C_kA_iB_j = \mathbf{C}\cdot(\mathbf{A}\times\mathbf{B})\,,$$

where we've used the properties of ε_{ijk} to prove a relation among triple products with the vectors in a different order.

go back to rules

- A very useful identity (if the repeated index is not first in both ϵ's, permute until it is):

$$\varepsilon_{ijk}\varepsilon_{ilm} = \delta_{jl}\delta_{km} - \delta_{jm}\delta_{kl}\,.$$

If repeated ε's, use this.

e.g. $\varepsilon_{123}\varepsilon_{113}=0$
only $\varepsilon_{123}\varepsilon_{123}$ or $\varepsilon_{123}\varepsilon_{132}$

j and l same and k and m same or with sign flip (be able to write this down.

4. **Example: Proving a Vector Identity**

- We'll write the i^{th} Cartesian component of the gradient operator $\boldsymbol{\nabla}$ as ∂_i (cf. $\frac{\partial}{\partial x_i}$).
- Let's simplify $\boldsymbol{\nabla}\times(\boldsymbol{\nabla}\times\mathbf{A}(\mathbf{x}))$. We start by considering the i^{th} component and then we use our expression for the cross product (working from the outside in):

$$(\boldsymbol{\nabla}\times(\boldsymbol{\nabla}\times\mathbf{A}))_i = \varepsilon_{ijk}\partial_j(\boldsymbol{\nabla}\times\mathbf{A})_k\,.$$

Next we replace the remaining cross product, making sure to introduce new dummy summation variables l and m:

$$(\boldsymbol{\nabla}\times(\boldsymbol{\nabla}\times\mathbf{A}))_i = \varepsilon_{ijk}\partial_j\varepsilon_{klm}\partial_lA_m = \varepsilon_{kij}\varepsilon_{klm}\partial_j\partial_lA_m\,.$$

(The partial derivatives act only on the components of $\mathbf{A}$, so we can pull out the ε's.) We rotated the indices in one of the ε's in the last step so that we can now directly apply our very useful identity (and simplify):

$$(\boldsymbol{\nabla}\times(\boldsymbol{\nabla}\times\mathbf{A}))_i = (\delta_{il}\delta_{jm} - \delta_{im}\delta_{lj})\partial_j\partial_lA_m = \partial_m\partial_iA_m - \partial_l\partial_lA_i = \partial_i(\partial_mA_m) - (\partial_l\partial_lA)_i$$

or, finally,

$$\boldsymbol{\nabla}\times(\boldsymbol{\nabla}\times\mathbf{A}) = \boldsymbol{\nabla}(\boldsymbol{\nabla}\cdot\mathbf{A}) - \nabla^2\mathbf{A}\,.$$

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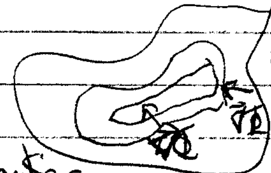
Div ($\vec{\nabla}\cdot\vec{A}$), Grad ($\vec{\nabla}\psi$), Curl ($\vec{\nabla}\times\vec{A}$) and all that...

- Look at Jackson covers handout for explicit forms of vector operations
- Read Arfken or Lea for more on physical interpretations

- gradient: $\vec{\nabla} = \hat{x}\frac{\partial}{\partial x} + \hat{y}\frac{\partial}{\partial y} + \hat{z}\frac{\partial}{\partial z}$ ⇐ derivatives give information about local change

scalar field $\Phi(x,y,z)$

$$\Rightarrow \vec{\nabla}\Phi = \hat{x}\frac{\partial\Phi}{\partial x} + \hat{y}\frac{\partial\Phi}{\partial y} + \hat{z}\frac{\partial\Phi}{\partial z}$$



← Φ = constant lines

Take small step in Cartesian coordinates

$$\Delta\vec{r} = \hat{x}\Delta x + \hat{y}\Delta y + \hat{z}\Delta z$$

How much does Φ change?

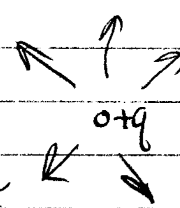
$$\Delta\Phi \doteq \frac{\partial\varphi}{\partial x}\Delta x + \frac{\partial\varphi}{\partial y}\Delta y + \frac{\partial\varphi}{\partial z}\Delta z = \vec{\nabla}\Phi\cdot\Delta\vec{r}$$

↙ 9 terms, 6 zero ($\hat{x}\cdot\hat{x}=1$, $\hat{x}\cdot\hat{y}=0$)

- If $\Delta\Phi = 0$, then "equipotential" traced by $\Delta\vec{r}$ steps, so $\vec{\nabla}\Phi \perp \Delta\vec{r}$
- Vector $\vec{\nabla}\Phi$ points in direction of most rapid spatial change

- divergence $\vec{V} = \hat{x}V_1 + \hat{y}V_2 + \hat{z}V_3$

$$\Rightarrow \vec{\nabla}\cdot\vec{V} = \frac{\partial V_1}{\partial x} + \frac{\partial V_2}{\partial y} + \frac{\partial V_3}{\partial z} \equiv \frac{\partial}{\partial x_i}V_i$$



- measures spreading out of vector fields around a point
- If $\vec{\nabla}\cdot\vec{V} = 0 \Rightarrow$ "solenoidal" [more physics in E+M!]

- curl $\vec{\nabla}\times\vec{V} = \hat{x}\left(\frac{\partial}{\partial y}V_3 - \frac{\partial}{\partial z}V_2\right) + \hat{y}\left(\frac{\partial}{\partial z}V_1 - \frac{\partial}{\partial x}V_3\right) + \hat{z}\left(\frac{\partial}{\partial x}V_2 - \frac{\partial}{\partial y}V_1\right)$

$$\equiv \varepsilon_{ijk}\frac{\partial}{\partial x_j}V_k$$ (or determinant representation)

- physically associated with circulation – integral of vector around closed curve. If $\vec{\nabla}\times\vec{V} = 0 \Rightarrow$ "irrotational"

← infinitesimal (very small!) loop

General: derivatives tell you about changes nearby

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Practical use with formulas

$$\vec{\nabla}\cdot\vec{x} = \left(\hat{x}\frac{\partial}{\partial x} + \hat{y}\frac{\partial}{\partial y} + \hat{z}\frac{\partial}{\partial z}\right)\cdot\left(\hat{x}x + \hat{y}y + \hat{z}z\right)$$

$$\overset{3\text{ terms}}{=} \frac{\partial x}{\partial x} + \frac{\partial y}{\partial y} + \frac{\partial z}{\partial z} = 3 \qquad \text{or } A_1 = x,\ A_2 = y,\ A_3 = z$$

$$\vec{\nabla}\times\vec{x} = \hat{x}\left(\cancel{\frac{\partial x}{\partial y}}^{0} - \frac{\partial y}{\partial x}\right) + \hat{y}\left(\cancel{\ }^{0}\right) + \hat{z}\left(\cancel{\ }^{0}\right) = 0$$

• Curvilinear coordinates → take advantage of geometry
(bring up page 29 from Lea notes Chap. 1 for cylindrical)

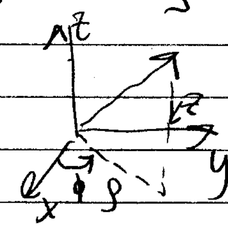
• General expressions in text such as (eg. front cover & Arfken)

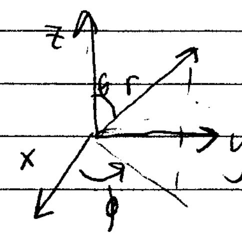
$$\vec{\nabla}\Phi = \sum_i \hat{q}_i \frac{1}{h_i}\frac{\partial\Phi}{\partial q_i} \qquad h_i \Rightarrow \text{scale factors}$$

[remember: it's a vector — common error]

$$\vec{\nabla}\cdot\vec{A} = \frac{1}{h_1h_2h_3}\left[\frac{\partial}{\partial q_1}\left(A_1h_2h_3\right) + \frac{\partial}{\partial q_2}\left(A_2h_1h_3\right) + \frac{\partial}{\partial q_3}\left(A_3h_1h_2\right)\right]$$

[Jackson $\hat{q}_i \to \hat{e}_i$]	$\hat{q}_1$	h_1	$\hat{q}_2$	h_2	$\hat{q}_3$	h_3
cartesian	$\hat{x}$	1	$\hat{y}$	1	$\hat{z}$	1
cylindrical	$\hat{\rho}$	1	$\hat{\phi}$	ρ	$\hat{z}$	1
spherical	$\hat{r}$	1	$\hat{\theta}$	r	$\hat{\phi}$	$r\sin\theta$





✳ • Need to carry h_i's in formulas (units, at least!)
for $\vec{\nabla}\Phi$, $\vec{\nabla}\cdot\vec{V}$, $\vec{\nabla}\times\vec{V}$

What can go wrong?

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Consider cylindrical coordinates

$$\vec{\nabla} = \hat{\rho}\frac{\partial}{\partial\rho} + \hat{\phi}\frac{1}{\rho}\frac{\partial}{\partial\phi} + \hat{z}\frac{\partial}{\partial z}$$ unit consistency

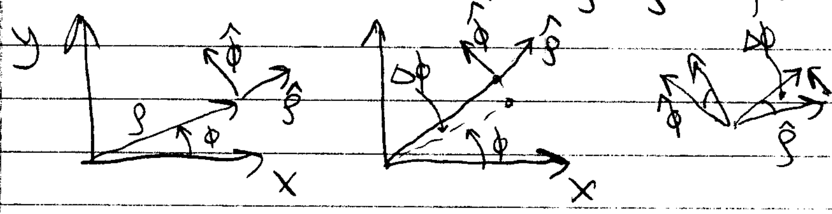
take $\vec{A} = \hat{\rho}A_1 + \hat{\phi}A_2 + \hat{z}A_3$

Now $\vec{\nabla}\psi = \hat{\rho}\frac{\partial\psi}{\partial\rho} + \hat{\phi}\frac{1}{\rho}\frac{\partial\psi}{\partial\rho} + \hat{z}\frac{\partial\psi}{\partial z}$

but $\vec{\nabla}\cdot\vec{A} \overset{?}{=} \frac{\partial A_1}{\partial\rho} + \frac{1}{\rho}\frac{\partial A_2}{\partial\phi} + \frac{\partial A_3}{\partial z}$ Wrong! disagrees with Jackson (has $\frac{1}{\rho}\frac{\partial}{\partial\rho}(\rho A_1) = \frac{\partial A_1}{\partial\rho} + \frac{A_1}{\rho}$)

Where did we fail?

$\Rightarrow$ Units vectors move! $\hat{\rho} = \hat{\rho}(\phi)$, $\hat{\phi} = \hat{\phi}(\phi)$, $\hat{z} = $ const.



$\Delta\hat{\rho} = \Delta\phi\,\hat{\phi}$ or $\frac{d\hat{\rho}}{d\phi} = \hat{\phi}$!

$$\Rightarrow \hat{\phi}\frac{1}{\rho}\frac{\partial}{\partial\phi}\left[\hat{\rho}A_1\right] = \hat{\phi}\frac{1}{\rho}\hat{\rho}\frac{\partial A_1}{\partial\phi} + \hat{\phi}\frac{1}{\rho}\hat{\phi}A_1 = \frac{1}{\rho}A_1 \checkmark$$

all other terms zero.

- Try it with spherical coordinates!
- Moral: It can be dangerous to get too used to Cartesian coordinates. (Or, Cartesian is safest! :))

Back to $\vec{\nabla}\cdot\vec{x}$. $\vec{x} = \hat{\rho}\rho + \hat{z}z \Rightarrow x_1 = \rho, x_2 = 0, x_3 = z$

$$\vec{\nabla}\cdot\vec{x} = \frac{1}{\rho}\frac{\partial}{\partial\rho}(\rho\cdot\rho) + \frac{1}{\rho}\frac{\partial}{\partial\phi}\cdot 0 + \frac{\partial}{\partial z}z = \frac{1}{\rho}2\rho + 1 = 3 \checkmark$$

You try $\vec{\nabla}\times\vec{x}$!

Spherical $\vec{x} = \hat{r}r$ that's it! Simple applications often (see HW).

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What about other formulas on Jackson covers?

$$\vec{\nabla}\cdot\left[\hat{r}f(r)\right] \underset{\substack{A_3=0\\ \text{spherical}}}{\overset{\substack{A_1=f(r)\\ A_2=0}}{=}} \frac{1}{r^2}\frac{\partial}{\partial r}\left(r^2 f(r)\right) = \frac{1}{r^2}2r\,f(r) + \frac{1}{r^2}r^2\frac{\partial f}{\partial r}$$

$$= \frac{2}{r}f(r) + \frac{\partial f}{\partial r} \quad \checkmark$$

What about Cartesian? $\vec{r} = \hat{r}r = \hat{x}x + \hat{y}y + \hat{z}z \Rightarrow (\vec{r})_i = x_i$

$(\hat{r})_i = \frac{x_i}{r}$

$$\vec{\nabla}\cdot\hat{r}f(r) \underset{\text{convention}}{\overset{\text{summation}}{=}} \frac{\partial}{\partial x_i}\left(\frac{x_i}{r}f(r)\right)$$

$$= \frac{\partial x_i}{\partial x_i}\frac{f(r)}{r} + x_i\frac{\partial \frac{1}{r}}{\partial x_i}f(r) + \frac{x_i}{r}\frac{\partial}{\partial x_i}f(r)$$

$$= \delta_{ii}\frac{f(r)}{r} + x_i\left(-\frac{1}{r^2}\right)\frac{\partial r}{\partial x_i}f(r) + \frac{x_i}{r}\frac{\partial f}{\partial r}\frac{\partial r}{\partial x_i}$$

$r^2 = x_i x_i$

$\Rightarrow 2r\frac{\partial r}{\partial x_i} = 2x_i$

$\Rightarrow \frac{\partial r}{\partial x_i} = \frac{x_i}{r}$ $\Rightarrow$

$$= 3\frac{f(r)}{r} - \frac{x_i x_i}{r^3}f(r) + \frac{x_i x_i}{r^2}\frac{\partial f}{\partial r} = \frac{2f(r)}{2} + \frac{\partial f}{\partial r} \quad \checkmark$$

<u>Homework!</u> $\vec{\nabla}\cdot\hat{r} \Rightarrow f(r)=1 \Rightarrow \vec{\nabla}\cdot\hat{r} = \frac{2}{r}$

$\vec{\nabla}\times\hat{\theta} \Rightarrow A_1=0, A_2=1, A_3=0$

$\Rightarrow \vec{\nabla}\times\hat{\theta} = \hat{r}\cdot 0 + \hat{\theta}\cdot 0 + \hat{\phi}\frac{1}{r}\frac{\partial}{\partial r}(r\cdot 1)$

$= \frac{1}{r}\hat{\phi}$

spherical

and so on.

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Vector Calculus Theorems → "Jackson covers" again

Divergence theorem $\int_V \vec{\nabla}\cdot\vec{A}\ d^3x = \int_S \vec{A}\cdot\hat{n}\ da$ ← "flux" of A through surface S

$\hat{n}$ is outward normal

$$\int_V \vec{\nabla}\psi\ d^3x = \int_S \psi\ \hat{n}\ da$$

$$\int_V \vec{\nabla}\times\vec{A}\ d^3x = \int_S \hat{n}\times\vec{A}\ da$$

Stokes's theorem: $\int_S (\vec{\nabla}\times\vec{A})\cdot\hat{n}\ da = \oint_C \vec{A}\cdot d\vec{r}$

$$\int_S \hat{n}\times\vec{\nabla}\psi\ da = \oint_C \psi\ d\vec{r}$$

and more.

☆ What you need to know: a) intuitive idea

PS#1 examples { b) how to interpret (V, S, $\hat{n}$, orientation)

c) how to apply in specific cases

Common features: • relates local quantities like $\vec{\nabla}\cdot\vec{A}$ to global quantities like flux through surface $\int_S \vec{A}\cdot\hat{n}\ da$

• sum of derivatives in interior is related to value on boundary

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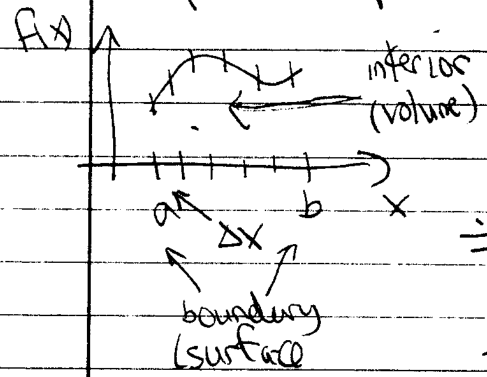
Simplest example: 1-d integrals of $f(x)$



$$\int_a^b \frac{df}{dx}\,dx = f(x)\Big|_{x=a}^{x=b} = f(b)-f(a)$$

$$\doteq \frac{f(a+\Delta x)-f(a)}{\Delta x}\Delta x + \frac{f(a+2\Delta x)-f(a+\Delta x)}{\Delta x}\cdot\Delta x$$

$$+\frac{f(a+3\Delta x)-f(a+2\Delta x)}{\Delta x}\Delta x + \dots$$

$$+\frac{f(b-\Delta x)-f(b-2\Delta x)}{\Delta x}\Delta x + \frac{f(b)-f(b-\Delta x)}{\Delta x}\Delta x$$

$$= f(b)-f(a)$$

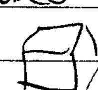
- everything cancels with adjacent contribution, leaving only boundaries uncanceled

$$\Rightarrow \int_V \vec{\nabla}\psi\, d^3x = \int_S \psi\,\hat{n}\,da \quad \text{component by component}$$

generic! E.g. $\int_V \vec{\nabla}\cdot\vec{A}\, d^3x = \int_S \vec{A}\cdot\hat{n}\,da$

derivative in interior — function at boundary

see text: break into cubes



$\vec{\nabla}\cdot\vec{A}$ in small cube $\propto$ outward flux.

But what goes out one cube goes in the next $\Rightarrow$ cancel except for ends

Similar for Stokes's Theorem. (see text)

Aside: vector dot product $\Delta\vec{x} = (\Delta x, \Delta x, \dots, \Delta x) \Rightarrow \Delta x_i$, $i=1,\#$ of divisions

$\vec{x} = (a, a+\Delta x, a+2\Delta x, \dots, b-\Delta x, b) \Rightarrow x_i$

$$\Rightarrow \int_a^b g(x)\,dx = \sum_i g(x_i)\Delta x_i \equiv g_i\,\Delta x_i$$

Do in parallel on computers. Continuous equations become finite matrix equations!

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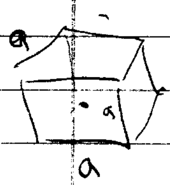
Example applications 1.2 and 1.3 from Lea text. (just setup)

1.2 Compute divergence of vector field $\vec{V} = k\vec{r}$, k = constant, $\vec{r}$ the position vector.

Find flux of $\vec{v}$ through cube of side a centered at origin and show it equals $\int_V \vec{\nabla}\cdot\vec{V}\, d^3x$ over volume.

Divergence: $\vec{\nabla}\cdot\vec{V} = k\vec{\nabla}\cdot(\hat{x}x+\hat{y}y+\hat{z}z) = k\left(\frac{\partial x}{\partial x}+\frac{\partial y}{\partial y}+\frac{\partial z}{\partial z}\right) = 3k$



$\Rightarrow$ volume integral is $3k\cdot\int_V d^3x = 3ka^3$

Flux: $\oint_{\text{surface}} k(x\hat{x}+y\hat{y}+z\hat{z})\cdot\hat{n}\, dA$ normals are $\hat{n} = \pm\hat{x}, \pm\hat{y}, \pm\hat{z}$

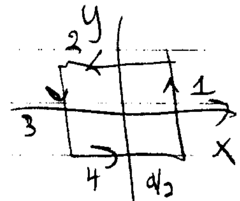
at $x=\frac{a}{2}$, $\hat{n}=+\hat{x}$ $\Rightarrow \int_{\text{face}} k\underbrace{x}_{=\frac{a}{2}}\,\hat{x}\cdot\hat{x}\; dA = k\frac{a}{2}\int_{\text{face}} dA = k\frac{a}{2}a^2 = \frac{ka^3}{2}$

at $x=\frac{a}{2}$, $\hat{n}=-\hat{x}$ but $\int_{\text{face}} k(-\frac{a}{2})\hat{x}\cdot\hat{x}\; dA = \frac{ka^3}{2} \xrightarrow[\text{same}]{\text{all}} 6\cdot\frac{ka^3}{2} = 3ka^3$

units? ✓

1.3 $\vec{u} = x^2y^3(\hat{x}+\hat{y})$

Find circulation of $\vec{u}$ around square of side a in x-y plane centered at origin. Compare $\int(\vec{\nabla}\times\vec{u})\cdot\hat{n}\, dA$



$\oint \vec{u}\cdot d\vec{r} = I_1+I_2+I_3+I_4 = 0 - \frac{2}{3}\left(\frac{a}{2}\right)^6 + 0 - \frac{2}{3}\left(\frac{a}{2}\right)^6 = -\frac{4}{3}\left(\frac{a}{2}\right)^6$ note sign

$d\vec{r} = -dy\,\hat{y}$ $\quad y<0$

$(\vec{\nabla}\cdot\vec{u})\cdot\underbrace{\hat{n}}_{\hat{z}} = (\vec{\nabla}\times\vec{u})_z = \frac{\partial u_y}{\partial x} - \frac{\partial u_x}{\partial y} = 2xy^3 - 3x^2y^2$

Surface integral $\int(\vec{\nabla}\times\vec{u})\cdot\hat{n}\, dA = \int_{-a/2}^{a/2}\int_{-a/2}^{a/2}(2xy^3-3x^2y^2)\,dy\,dx = \int_{-a/2}\left(\frac{xy^4}{2}-x^2y^3\right)\Big|_{-a/2}^{a/2} dx$

$= 2\int_{-a/2}^{a/2} -x^2\left(\frac{a}{2}\right)^3 dx = -2\left(\frac{a}{2}\right)^3\frac{x^3}{3}\Big|_{-a/2}^{a/2} = -\frac{4}{3}\left(\frac{a}{2}\right)^6$ ✓

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Partial integration in vector calculus

recall $\int_a^b u\,dv = uv\Big|_a^b - \int_a^b v\,du$

or $\int_a^b (u\,dv + v\,du) = \int_a^b d(uv) = uv\Big|_a^b$

volume surface

or $\int_a^b u(x)\frac{dv(x)}{dx} = -\int_a^b v(x)\frac{du(x)}{dx} + u(x)\,v(x)\Big|_a^b$

$\Rightarrow$ switch $\frac{d}{dx}$ from $v(x)$ to $u(x)$ costs minus sign plus surface term (which often vanishes)

cf. Green's identity and theorem

$$\int_V \phi\,\vec{\nabla}^2\psi\,d^3x \underset{\vec{\nabla}}{\overset{\text{move}}{=}} -\int_V \vec{\nabla}\phi\cdot\vec{\nabla}\psi\,d^3x + \int_S \phi(\vec{\nabla}\psi)\cdot\hat{n}\,da$$

$$-\int_V \vec{\nabla}\phi\cdot\vec{\nabla}\psi\,d^3x \underset{\vec{\nabla}}{\overset{\text{move}}{=}} +\int_V (\vec{\nabla}^2\phi)\psi\,d^3x - \int_S (\vec{\nabla}\phi)\psi\cdot\hat{n}\,da$$

$$\Rightarrow \int_V \phi\,\vec{\nabla}^2\psi\,d^3x = \int_V (\vec{\nabla}^2\phi)\psi\,d^3x + \int_S \left[\phi(\vec{\nabla}\psi) - (\vec{\nabla}\phi)\psi\right]\cdot\hat{n}\,da$$

as on the cover.