10/26/11

834 Lecture 10

Lecture plan:

- Follow-ups to Fourier series from last time
  - including Mathematica
  - convergence of Fourier series 88
- Introduction to generalized function (survey)
  - core competencies
  - typical problems
- Solving S. eqn for bound states (Numerov method) 68-70
  - finally?

Before class:

- Start up IE with 834 page. Start Mathematica with Fourier_series2.nb, Fourier_series3.nb, and ps5_checks.nb
  - Start up the animation.

On board:

- Summary results for Fourier series
- 3 series defined on $0 \le x \le L$

use for damped harmonic oscillator with periodic driving force

1. Full series
$$f(x) = \sum_{n=-\infty}^{\infty} c_n e^{in2\pi x/L}; \quad c_n = \frac{1}{L}\int_0^L f(x)\, e^{-in2\pi x/L} \quad \text{and period } L$$
$$= b_0 + \sum_{n=1}^{\infty}\left(a_n \sin\frac{n2\pi x}{L} + b_n \cos\frac{n2\pi x}{L}\right) \quad \text{where } a_n = i(c_n - c_{-n}),\ b_n = (c_n + c_{-n}),\ b_0 = c_0$$

use for diff. eqs. with matching boundary conditions

2. Sine series: $f(x) = \sum_{n=1}^{\infty} a_n \sin\frac{n\pi x}{L}$; $a_n = \frac{2}{L}\int_0^L f(x)\sin\frac{n\pi x}{L}\,dx$, period $2L$

3. cosine series: $f(x) = b_0 + \sum_{n=1}^{\infty} b_n \cos\frac{n\pi x}{L}$; $b_n = \frac{2}{L}\int_0^L f(x)\cos\frac{n\pi x}{L}\,dx$, $b_0 = \frac{1}{L}\int_0^L f(x)\,dx$; period $2L$

Start with the summary of 3 series and then Fourier_series3.nb

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Lea problem 22: Generalized Parseval Theorem:
If $f(x)$ is represented by the series $\sum_n f_n e^{inx}$ over the interval $0 < x < 2\pi$ and $g(x) = \sum_n g_n e^{inx}$ over the same range, prove that (for f and g real)

$$\frac{1}{2\pi}\int_0^{2\pi} f^*(x)\, g(x)\, dx = \sum_{n=-\infty}^{\infty} f_n^* g_n$$

Proof:
Substitute expansions in to the integral

$$\frac{1}{2\pi}\int_0^{2\pi}\left(\sum_n f_n^* e^{-inx}\right)\left(\sum_m g_m e^{imx}\right) dx$$

assume we can interchange sums and integrals

$$= \sum_n \sum_m f_n^* g_m \frac{1}{2\pi}\underbrace{\int_0^{2\pi} e^{-inx} e^{imx}\, dx}$$

$$\frac{1}{i(m-n)} e^{i(m-n)x}\Big|_0^{2\pi} \quad \text{if } m \neq n$$

$$x\Big|_0^{2\pi} \quad \text{if } m = n$$

$$\underbrace{\frac{1}{2\pi}\frac{1}{i(m-n)}\left(e^{i(m-n)2\pi} - e^0\right)}_{} \quad \text{if } m \neq n$$

0, integer

$$2\pi \quad \text{if } m = n$$

$$= \sum_n \sum_m f_n^* g_m \delta_{mn}$$

$$= \sum_n f_n^* g_n \qquad \text{QED}$$

Suppose $f(x) = g(x) \Rightarrow \frac{1}{2\pi}\int_0^{2\pi} |f(x)|^2 dx = \sum_{n=-\infty}^{\infty} |f_n|^2$

cf. normalization and coefficients of basis in quantum mechanics.

More abstract $\langle f|g\rangle = \frac{1}{2\pi}\int_0^{2\pi} \langle f|x\rangle\langle x|g\rangle dx = \sum_n \langle f|\phi_n\rangle\langle\phi_n|g\rangle$

$\uparrow \frac{1}{2\pi}\int_0^{2\pi} |x\rangle\langle x|\, dx$ $\qquad \uparrow \sum_n |\phi_n\rangle\langle\phi_n|$ $\qquad \uparrow f_n^*$ $\qquad \uparrow g_n$

10/26/11

Consider Lea problem 4.4:

An odd function f(x) on the range (-L, L) has the additional property that $f(x+L) = -f(x)$

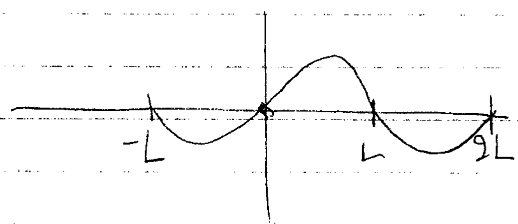
a) make a sketch showing important features of the function

b) which kind on Fourier series (sine, cosine, or full) represents this function on the range $-L \leq x \leq L$?

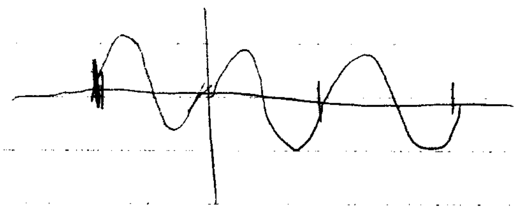
c) Show that the series has only terms of odd order ($n = 2m+1$) and find a formula for the coefficients as an integral over $0 \leq x \leq L/2$.

d) Repeat with $f(x+L) = +f(x)$

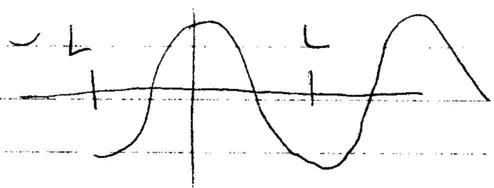
e) Repeat with even function and $f(x+L) = -f(x)$.

a) 

b), c) sine series with only odd terms since even about $x = \frac{L}{2}$
$\Rightarrow$ determined by $[0, \frac{L}{2}]$ integral

d) 

$f(x+L) = +f(x)$
$\Rightarrow$ sine series with only <u>even</u> terms

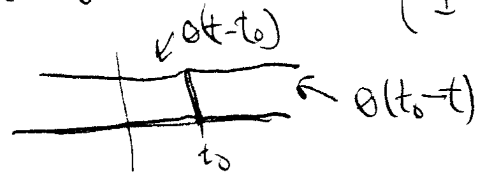
e) 

$f(x+L) = -f(x)$ but even
$\Rightarrow$ cosine series with odd terms
(to about $f(x+L) = f(x)$)

Theta function follow up:

$$\theta(t-t_0) = \begin{cases} 1 & t>t_0 \\ 0 & t<t_0 \end{cases} \qquad \theta(t_0-t) = \begin{cases} 0 & t>t_0 \\ 1 & t<t_0 \end{cases}$$

(What is $t=t_0$?)



Note $\theta(t-t_0) + \theta(t_0-t) = 1$ for any $t, t_0 \Rightarrow$ suggests $\theta(0) = \frac{1}{2}$.

What is $\frac{d}{dt}\theta(t-t_0)$? (Is $\theta(t)$ a well-defined function?)

Consider $\displaystyle\int_{-\infty}^{\infty} f(t) \frac{d}{dt}\theta(t-t_0)\, dt$ where $f(t) \xrightarrow[|t|\to\infty]{} 0$

Then partially integrate $\xRightarrow[\text{surface terms vanish}]{\text{minus sign}}$ $\displaystyle = -\int_{-\infty}^{\infty} \frac{df}{dt}\theta(t-t_0) = -\int_{t_0}^{\infty} \frac{df}{dt}\, dt$

$$= -[f(\infty) - f(t_0)] = f(t_0)$$

Arbitrary $f(t)$, $\Rightarrow \frac{d}{dt}\theta(t-t_0) = \delta(t-t_0)$!

10/26/11

Generalized Functions: Core Competencies

① Applying properties of delta/theta functions
② Identification of delta functions in physics context } in appropriate coordinates
③ Fourier representations
④ Differential equations with impulse terms

• Here focus on intuition and applications rather than proofs and mathematical formalism of distributions.

① Applying properties of delta/theta functions (sometimes the 3 is omitted)

• These functions are idealization of physical situations
  • charge density of point charge at $\vec{x}_0$: $\rho(\vec{x}) = q\,\delta^3(\vec{x}-\vec{x}_0)$
    [notation $\delta^3(\;)$ as reminder $\delta(x)\delta(y)\delta(z)$ in Cartesian]
  • impulse driving force at t_0: $F(t) = C\,\delta(t-t_0)$;
    e.g. strike string or rod at $t=t_0$ abruptly
  • Start applying a driving force at $t=t_0$:
    $F(t) = [F_0 \sin \omega t]\,\Theta(t-t_0)$

⇒ in the real world, charges are spread out at least a little (or, as for electrons, we can't tell), forces take some time to deliver or start up.
  • So it is natural physically and not just mathematically to consider delta sequences. Prove properties for delta sequences!
  • In field theory: "regularized" delta functions

$$\lim_{n\to\infty} \phi_n(x) = \delta(x)$$

PS#6
$$\phi_n^{(1)}(x) = \frac{n}{\pi}\left(\frac{1}{1+n^2x^2}\right)$$

$$\phi_n^{(2)}(x) = \frac{1-\cos nx}{n\pi x^2}$$

• Look at these with Mathematica: ps6_checks.nb

10/26/11

Need to demonstrate "sifting property"

check This works for $\{\phi_n\}$ → in $n\to\infty$ limit (PS#6: contour integration)

$$\int_{-\infty}^{\infty} f(x)\,\delta(x)\,dx = f(0) \qquad \left[\int_{-\infty}^{\infty} f(x)\,\delta(x-a)\,dx = f(a)\right]$$

from change of variables on delta sequence

[sift out the value at $x=0$ of the test function $f(x)$]

- restrictions on $f(x)$? Depends on ϕ_n sequence
  - we usually require $\int_{-\infty}^{\infty} |f(x)|^2\,dx < \infty \Rightarrow$ "square integrable"

  For physics reasons. This is usually sufficient.
  - PS#6: does $f(x)$ have to be analytic for delta sequence?

Another property: $\delta(ax) = \frac{1}{|a|}\delta(x)$

- such an equation is always a shorthand because we'll be applying it in an integral
- Would see to follow from change of variables

$$\int_{-\infty}^{\infty} f(u)\,\delta(au)\,du \underset{dx=a\,du}{\overset{x=au}{=}} \int_{-\infty}^{\infty} f\!\left(\tfrac{x}{a}\right)\delta(x)\,\tfrac{1}{a}\,dx = \tfrac{1}{a}f(0) \overset{?}{=} \int_{-\infty}^{\infty} f(x)\,\tfrac{1}{a}\,\delta(x)\,dx \quad ??$$

But where is $|a|$? We have implicitly assumed $a>0$.
If $a<0$, we also have to flip the limits

$$a<0:\ \int_{+\infty}^{-\infty} f\!\left(\tfrac{x}{a}\right)\delta(x)\,\tfrac{1}{a}\,dx = -\tfrac{1}{a}\int_{-\infty}^{\infty} f\!\left(\tfrac{x}{a}\right)\delta(x) = -\tfrac{1}{a}f(0) \overset{\text{general}}{\Rightarrow} \tfrac{1}{|a|}\delta(x)$$

- What else? Derivatives! (prove with delta sequences. Here: mnemonic)

$$\int_{-\infty}^{\infty} \delta'(x)\,f(x)\,dx = \int_{-\infty}^{\infty}\left[\tfrac{d}{dx}\delta(x)\right]f(x)\,dx = -\int_{-\infty}^{\infty}\delta(x)\,\tfrac{d}{dx}f(x)\,dx + \text{surface}$$

switching derivative; surface since $\delta(\pm\infty)=0$

$$\Rightarrow \int_{-\infty}^{\infty}\delta'(x)\,f(x)\,dx = -f'(0) \ \underset{\text{keep doing it}}{\Rightarrow}\ \int_{-\infty}^{\infty}\delta^{(n)}(x)\,f(x)\,dx = (-1)^n f^{(n)}(0)$$

switch sign every time

looks like Taylor series coefficient!

10/26/11

- delta function of a function. Prove with delta sequences. Result

$$\int_{-\infty}^{\infty} \delta(g(x))\, f(x)\, dx = \sum_{i=1}^{N} \frac{f(x_{0_i})}{|g'(x_{0_i})|}$$

x_{0_i} are N zeros of $g(x)$: $g(x_{0_i}) = 0$, $i = 1, \dots, N$

eg. PS# 6 $\int_{-\infty}^{\infty} e^{-x^2} \delta(x^2+x-6)\, dx$

What is the idea?

Suppose $g(x) = x^2 - (a+b)x + ab = (x-a)(x-b)$

Near $x=a$, variation of $x-b$ part is slow so replace by $a-b$

$$\Rightarrow \delta((x-a)(x-b)) \xrightarrow[x=a]{\text{near}} \delta((a-b)(x-a)) = \frac{1}{|a-b|}\delta(x-a) \qquad a = x_{0_1}$$

← from $\delta(cx) = \frac{1}{|c|}\delta(x)$

$$\xrightarrow[x=b]{\text{near}} \delta((b-a)(x-b)) = \frac{1}{|b-a|}\delta(x-b) \qquad b = x_{0_2}$$

$$\Rightarrow \int_{-\infty}^{\infty} \delta((x-a)(x-b))\, f(x)\, dx = \frac{1}{|a-b|} f(a) + \frac{1}{|b-a|} f(b)$$

Note that $|g'(a)| = |2x - (a+b)x|\Big|_{x=a} = |2a - (a+b)| = |a-b|$

and $|g'(b)| = (b-a) = |a-b|$.

- More generally, near $x = x_{0_i}$, $g(x) = \cancel{g(x_{0_i})} + (x - x_{0_i}) g'(x_{0_i})$ ← determines behavior

0 by construction

$+ \frac{1}{2}(x - x_{0_i})^2 g''(x_{0_i}) + \dots$

why can we neglect? (look at $\{\phi_n\}$'s)

What if there is a double (or higher) root?

Eg. $(x-a)^2$?

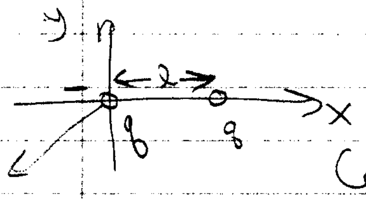
Then the formula does not apply!

10/26/11

⑧ Identification of delta functions in physics contexts

potential from charge density $\Phi(\vec{x}) = \frac{1}{4\pi\varepsilon_0}\int \frac{\rho(\vec{x}')}{|\vec{x}-\vec{x}'|}\, d^3x$

Suppose dipole: point charges q at separation l



$\rho(\vec{x}) = -q\,\delta(x)\,\delta(y)\,\delta(z) + q\,\delta(x-l)\,\delta(y)\,\delta(z)$

Coefficient correct? Check total charge from + charge

$$q \overset{?}{=} \int_{-\infty}^{\infty} dx \int_{-\infty}^{\infty} dy \int_{-\infty}^{\infty} dz\; q\,\delta(x-l)\,\delta(y)\,\delta(z)\, dx\,dy\,dz = q \;\checkmark$$

What about <u>ideal</u> dipole? $l \to 0$, $\vec{p} = p\hat{x}$, $p = ql$ finite as $l \to 0$

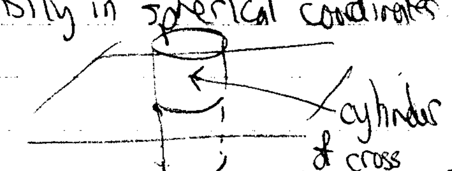
$$\Rightarrow \rho(\vec{x}) \xrightarrow[l\to 0]{} -q\,\delta(y)\,\delta(z)\left[\delta(x) - \delta(x-l)\right]$$

$$= \lim_{l\to 0} -\underbrace{ql}_{p}\,\delta(y)\,\delta(z)\left[\frac{\delta(x)-\delta(x-l)}{l}\right] = -p\,\delta(y)\,\delta(z)\,\delta'(x)$$

derivative of delta function!

Example 6.2 in Lea: (more on PS#6!)

- Sheet of charge in $z=0$ plane with surface charge density σ_0 (charge/area). Find volume charge density in spherical coordinates



Only at $z=0 \Rightarrow \rho(\vec{x}) = k\,\delta(z)$. What is k?

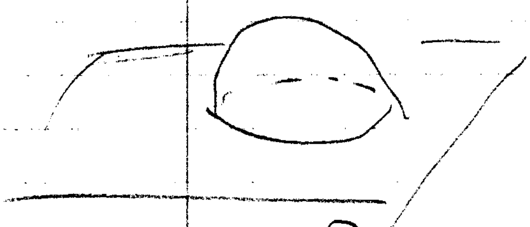
find charge in cylinder (should be $\sigma_0\, dA$) $dq = \int_{\text{cylinder}} \rho(\vec{x})\, dV = \left(\int_{-\infty}^{\infty} k\,\delta(z)\, dz\right) dA = k\, dA$
$= \sigma_0\, dA \Rightarrow \boxed{k = \sigma_0}$

- change z to spherical $\rho(\vec{x}) = \sigma_0\,\delta(r\cos\theta)$ ← only zero in $0 \le \theta \le \pi$

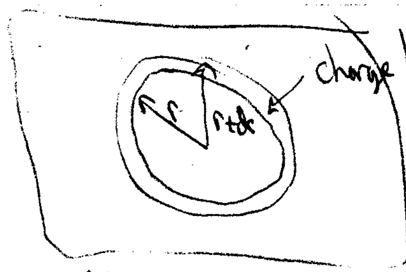
$$\underset{\delta(cx) = \frac{1}{|c|}\delta(x)}{=} \frac{\sigma_0\,\delta(\cos\theta)}{r} \underset{\text{find zeros in } 0\le\theta\le\pi}{=} \frac{\sigma_0}{r}\frac{\delta(\theta - \frac{\pi}{2})}{\left[-\sin\theta\right]_{\theta=\frac{\pi}{2}}} = \frac{\sigma_0}{r}\,\delta\!\left(\theta - \frac{\pi}{2}\right)$$

10/26/11

How to check our result? Integrate in spherical coordinates over shell & thickness dr



From above



charge here is $\sigma_0 dA$
$= \sigma_0 2\pi r\, dr$

$$dq = \int_{shell} \rho(\vec{x})\, dV = \int_0^{2\pi} d\phi \int_0^{\pi} d\theta\, \frac{\sigma_0}{r}\delta(\theta - \tfrac{\pi}{2})\, r^2 \sin\theta\, dr$$

$$= 2\pi \frac{\sigma_0}{r} r^2 \sin\tfrac{\pi}{2}\, dr = \sigma_0 2\pi r\, dr \ \checkmark$$

③ Fourier representations

For $-L < x < L$ $\quad \phi_n(x) = \sum_{m=-\infty}^{\infty} c_m^{(n)} e^{im\pi x/L}$

Find c_m for given $\{\phi_n\}$ and then take limit

Result $c_m^{(n)} \xrightarrow[n\to\infty]{} \frac{1}{2L} \Rightarrow \delta(x) = \frac{1}{2L}\sum_{m=-\infty} e^{im\pi x/L}$

* All Fourier modes contribute with the same amplitude!
.every harmonic is equal.

$\xrightarrow[\text{Transform}]{\text{Fourier}}$ $\boxed{\delta(t) = \frac{1}{2\pi}\int_{-\infty}^{\infty} e^{i\omega t}\, d\omega}$

cf. $\langle x|x'\rangle = \delta(x-x') = \sum_{m=-\infty}^{\infty} \underbrace{\langle x|m\rangle}_{\frac{1}{\sqrt{2L}}e^{im\pi x/L}} \underbrace{\langle m|x'\rangle}_{\frac{1}{\sqrt{2L}}e^{-im\pi x'/L}}$

(are $\frac{1}{\sqrt{2}}$'s correct?)

④ Differential equations with impulse terms
next time!: wave equation (string!) with impulse excitation