10/31/11

834 Lecture 11

Lecture Plan

- Comments on PS#5
- Continuation of delta function properties and application
  - focus on problems
- Probably won't get to this! → - Solving S-eqn for bound states (Numerov method) 68-70

Before class:

- return PS#5 up front (if available, otherwise at break)
- start up 834 page + Mathematica diracdelta.nb
- Try out DOSBox and CUPS Waves and Optics

On board:

- Comments on PS#5
  - be careful of constant term ⇒ $n=0$. Don't include it in sums over sines and cosines (which should start at $n=1$) Be sure to calculate it!
  - Choosing when to use exponentials: any time you mix sines and cosines, as when $x(t)$ and $\dot{x}(t)$, exponentials are much easier
  - Projecting coefficients of $f(x) = \sum_{n=-\infty}^{\infty} c_n e^{inx}$ should integrate over e^{-inx} ← minus sign
- Summary of delta function properties
  - $\int_{-\infty}^{\infty} f(x)\,\delta(x-x_0)\,dx = f(x-x_0)$ "sifting property" $\int_{-\infty}^{\infty} |f(x)|^2\,dx < \infty$ "square integrable" usually required
  - $\delta(ax) = \frac{1}{|a|}\delta(x)$ for constant a
  - $\delta'(x) f(x) = -\delta(x) f'(x) \Rightarrow \int_{-\infty}^{\infty} \delta'(x)\, f(x)\, dx = -f'(0)$
  - $\int \delta(g(x))\, f(x)\, dx = \sum_{i=1}^{N} \frac{f(x_{0i})}{|g'(x_{0i})|}$ x_{0i} are N zeros of $g(x)$.

10/31/11

Setting up DOSBox to run CUPS on Windows 7

- According to online documentation, it should work on both 32-bit and 64-bit versions.
- Google DOSBox and go to downloads page.
  - Download win32 installer to desktop
  - for options, install onto the desktop on a disk

Special function keys:

alt-enter	toggle full-screen
alt-pause	pause
ctrl-f9	Kill (close) DOS Box
ctrl-f10	capture/release the mouse
ctrl-f11	slow down
ctrl-f12	speed up

mount c d:\cupswo — mount the directory as the C drive
c: — change to the C drive
C:\> set TEMP = C:\TEMP
C:\> cupswo

- Fourier
  - File → Configuration
  - fourier → Display setup ⇒ change harmonics to 80 points to 400

- May work fine in small box but have trouble full screen

- Try Square Wave, Sawtooth Wave, Full-wave Rect, Pulse
  - For pulse, note width of pulse vs. coefficients needed (use zero as indication)

10/31/11

Delta function property follow-ups: (open dirac-delta.nb)

What if there is a double (or higher) root of $g(x)$?

Then $\int \delta(g(x))\, f(x)\, dx = \sum_{i=1}^{N} \frac{f(x_{0_i})}{|g'(x_{0_i})|}$ does not apply.

Quick check for class:

What is $\int_{-\infty}^{\infty} f(x)\, \delta(x^2-9)\, dx \overset{?}{=}$ $\quad g(x) = x^2-9 \Rightarrow x_{0_1} = 3$, $x_{0_2} = -3$

$\Rightarrow\ = \frac{f(3)}{6} + \frac{f(-3)}{6}$ $\quad \boxed{\text{or } \delta(x^2-9) = \frac{1}{6}\delta(x-3) + \frac{1}{6}\delta(x+3)}$

What is $\int_{-\infty}^{\infty} f(x)\, \delta(x^2+9)\, dx \overset{?}{=}$ ans. 0! Only real roots matter here!

Can we do a Dirac delta function in Mathematica?

Yes! $\delta(x) \to$ DiracDelta[x]

- Demonstrate sifting property, and $\delta(x-a)$

Try DiracDelta[x^2-9] // FunctionExpand

$\frac{1}{6}$ DiracDelta[-3+x] + $\frac{1}{6}$ DiracDelta[3+x]

good command to know; tries to expand special functions to simplify them

Predict DiracDelta[x^2+9] // FunctionExpand

ans: 0!

Try an Integrate with DiracDelta[x] to show sifting function

Try doing a double root [eg. $\delta((x-3)^2)$ or just $\delta(x^2)$]

- analytically in Mathematica (fails) then numerically (blows up)

Use $\lim_{\varepsilon\to 0} \frac{1}{\pi}\frac{\varepsilon}{x^2+\varepsilon^2} = \delta(x)$ [can you write this as a delta sequence? $\varepsilon = 1/n$?]

$\Rightarrow \int \delta(x^2)\, f(x)\, dx = \lim_{\varepsilon\to 0} \int_{-\infty}^{\infty} \frac{1}{\pi}\frac{\varepsilon}{x^4+\varepsilon^2} f(x)\, dx \overset{u=x^2}{=} \lim_{\varepsilon\to 0} \int_{-\infty}^{\infty} \frac{1}{\pi}\frac{\varepsilon}{u^2+\varepsilon^2}\, 2\, f(\sqrt{u})\frac{du}{2\sqrt{u}} = \int_{-\infty}^{\infty} \delta(u)\, \frac{f(\sqrt{u})}{2\sqrt{u}}\, du \to \infty!$

10/31/11

Comments on PS#6 problem 2: delta sequences

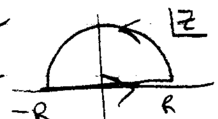
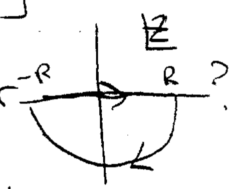
Part a). We can write $\phi_n(x)$ two ways:

$$\phi_n(x) = \frac{n}{\pi}\frac{1}{1+n^2x^2} = \frac{1}{n\pi}\frac{1}{x^2+1/n^2}$$

Which form do you think will be most useful for doing a contour integral and taking $n \to \infty$? (2nd one!)

The general idea is

$$\lim_{n\to\infty}\left[\int_{-\infty}^{\infty} dx\, \phi_n(x) f(x)\right] = \lim_{n\to\infty}\left[\oint_C dz\, \phi_n(z) f(z)\right]$$

for a suitable contour. Do we close  or  ?
(or something else?)

- Plan: assume $f(z)$ is analytic first and then relax that assumption as much as possible.

Does $\phi_n(z) = \frac{1}{n\pi}\frac{1}{z^2+1/n^2}$ care which half-plane the contour is closed in?

- no, since $1/z^2$, either way is equally good to make the integral over C_R vanish if $f(z)$ doesn't blow up.
- If $f(z)$ has a e^{iaz} factor, close according to the sign of $a \Rightarrow$ check that the answer doesn't depend on that sign!
- What other limitations are on $f(z)$ as $z \to \infty$? (recall Jordan's lemma, for example).
- When you have your first result assuming $f(z)$ analytic, then ask: what if $f(z)$ has a pole in the upper half plane and we close there? Does it matter? [What will the weighting of the pole be as $n \to \infty$?]

10/31/11

Part b) Now we apply contour integration again but with

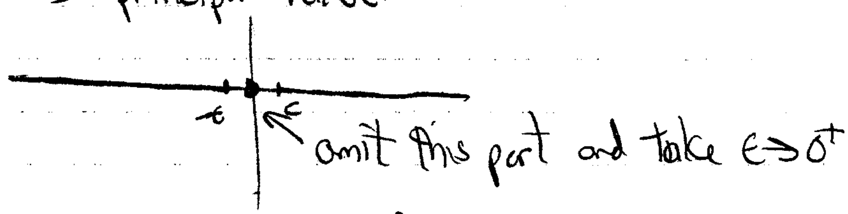
$$f(z) = \frac{1-\cos nz}{n\pi z^2}$$

Does $f(z)$ have a simple or double pole?

- Be careful: check the Laurent series near $z=0$ (expand numerator in Taylor series and look for non-zero $1/z$ coefficient in $f(z)$ expansion).

- Can we close $\oint f(z)\,dz$ in the upper-half plane? lower-half?
- How can we rewrite $1-\cos nz$?
  - Real part of exponential
  - two exponentials

- If we introduce a pole (or poles) as intermediate steps, we don't want to include that contribution in the integral $\Rightarrow$ principal value



- Evaluate principal value as in past examples, problems.

10/31/11

Fourier representations:

Claim: $\delta(x) = \frac{1}{2L} \sum_{m=-\infty}^{\infty} e^{im\pi x/L}$ on $-L<x<L$

Key feature: all of the c_m coefficients are the same constant!
$\Rightarrow$ every harmonic contributes equally.

Note that this sum is quite ill-defined, but we don't intend to use it in this form, but in an integral.

- Lea has an extended discussion of how to make this well defined, but we won't consider this now.

Where does this come from?

On $-L<x<L$, use the Fourier series of a delta sequence:

$$\phi_n(x) = \sum_{m=-\infty}^{\infty} C_m^{(n)} e^{im\pi x/L}$$

Plan: find c_m for given $\{\phi_n\}$ and then take $n\to\infty \Rightarrow C_m^{(n)} \xrightarrow[n\to\infty]{} \frac{1}{2L}$

** Try this out in Mathematica with explicit example and also build $\delta(x)$ directly. $\Rightarrow$ dirac_delta.nb

Check the sifting property (in the limit $n\to\infty$):

$$\int_{-L}^{L} \delta(x) f(x)\,dx = \int_{-L}^{L} \frac{1}{2L} \sum_{m=-\infty}^{\infty} e^{im\pi x/L} f(x)\,dx$$

$$\overset{\text{exchange order}}{=} \sum_{m=-\infty}^{\infty} \frac{1}{2L} \underbrace{\int_{-L}^{L} e^{im\pi x/L} f(x)\,dx}$$

But if $f(x) = \sum_{n=-\infty}^{\infty} C_n e^{in\pi x/L}$, what coefficient is projected out? Ans: c_{-m}

$$\Rightarrow \quad = \sum_{m=-\infty}^{\infty} c_{-m} = \sum_{m=-\infty}^{\infty} c_m = f(0) \; !$$

10/31/11

When we go from $-L < x < L$ to infinite intervals, then we have the Fourier transform representation:

$$\delta(t) = \frac{1}{2\pi}\int_{-\infty}^{\infty} e^{i\omega t}\, d\omega$$ same with $e^{-i\omega t}$ (why?)

We can find other representations, e.g.,

$$\delta(t) = \frac{1}{2\pi}\int_{-\infty}^{\infty} e^{i\omega t}\, d\omega = \frac{1}{2\pi}\left(\int_{-\infty}^{0} e^{i\omega t}\, d\omega + \int_{0}^{\infty} e^{i\omega t}\, d\omega\right)$$

$$= \frac{1}{2\pi}\left(\int_{\infty}^{0} e^{-i\omega' t}(-d\omega') + \int_{0}^{\infty} e^{i\omega t}\, d\omega\right)$$

$$= \frac{1}{2\pi}\left(\int_{0}^{\infty} (e^{-i\omega t} + e^{i\omega t})\, d\omega\right)$$

$$= \frac{1}{\pi}\int_{0}^{\infty} \cos(\omega t)\, d\omega$$

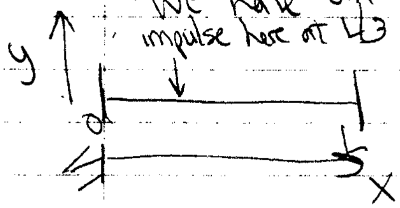
Similarly, $\delta(x-a) = \frac{2}{\pi}\int_{0}^{\infty} (\cos kx)(\cos ka)\, dk$

More later!

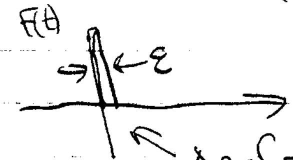
10/31/11

Consider example 6.3 to Lea.

- We have an initially stationary string that we hit with a hammer at $t=0$ with <u>impulse</u> I_0.



So force



$\Delta p = \int F(t)\,dt = I_0$ impulse!

- We consider the limit $\varepsilon \to 0$ and also that the impulse is delivered just at $x = L/3$.

- The wave equation is $\dfrac{\partial^2 y}{\partial x^2} = \dfrac{1}{v^2}\dfrac{\partial^2 y}{\partial t^2}$

and we solve by separation of variables with the fixed boundary conditions in x given by $y(0,t) = y(L,t) = 0$.

$$\text{So} \quad y(x,t) = \sum_{n=1}^{\infty} \sin\frac{n\pi x}{L}\left(a_n \sin\frac{n\pi v t}{L} + b_n \cos\frac{n\pi v t}{L}\right) \quad (+b_0 \text{ term})$$

In this case, $y(x,0) = 0$ for all $x \Rightarrow b_n \equiv 0$.

$$\Rightarrow \quad y(x,t) = \sum_{n=1}^{\infty}\left(\sin\frac{n\pi x}{L}\right) a_n \sin\left(\frac{n\pi v t}{L}\right)$$

$$\frac{\partial y}{\partial t}(x,t) = \sum_{n=1}^{\infty}\left(\sin\frac{n\pi x}{L}\right)\frac{n\pi v}{L}\, a_n \cos\left(\frac{n\pi v t}{L}\right)$$

What is the initial condition on $\dfrac{\partial y}{\partial t}$?

We apply an impulse to length dx, which has mass $\mu\, dx$ and so the change in momentum is

$$(\mu\, dx)\left(\frac{\partial y(x,0^+)}{\partial t} - \underbrace{\cancel{\frac{\partial y(x,0^-)}{\partial t}}}_{\text{starts at rest}}\right) = C\delta\left(x - \frac{L}{3}\right)dx$$

When we integrate over x, the total impulse is $I_0 \Rightarrow C = I_0$.

10/31/11

So $\frac{\partial y}{\partial t}(x, 0^+) = \frac{I_0}{\mu}\delta(x - \frac{L}{3})$ (check units!)

At $t_0 = 0^+$, $\frac{I_0}{\mu}\delta(x - \frac{L}{3}) = \sum_{n=1}^{\infty} \left(a_n \frac{n\pi v}{L}\right) \sin\frac{n\pi x}{L}$ ← coefficient. Project by multiplying by $\sin\frac{n\pi x}{L}$ and integrating

project $\Rightarrow a_n \frac{n\pi v}{L} = \frac{2}{L}\int_0^L \frac{I_0}{\mu}\delta(x - \frac{L}{3}) \sin\frac{n\pi x}{L}\,dx$

$$= \frac{2I_0}{L\mu}\sin\frac{n\pi}{3}$$

$\Rightarrow a_n = \frac{2}{n\pi v}\frac{I_0}{\mu}\sin\frac{n\pi}{3}$ decreases with n like $\frac{1}{n}$

$$y(x,t) = \sum_{n=1}^{\infty} \frac{2}{n\pi v}\frac{I_0}{\mu}\sin\frac{n\pi}{3}\sin\frac{n\pi x}{L}\sin\frac{n\pi v t}{L}$$

If $n = 3m$, this is zero $\Rightarrow$ third harmonics are missing.

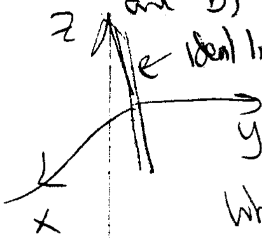
- Try this out in Mathematica!

10/31/11

Physical quantities (see 97 - 98)

Problem 6.17 in Lea

A line of charge with uniform line charge density λ (units charge/length) lies along the z-axis. Find the volume charge density a) in cylindrical coordinates and b) in spherical coordinates.



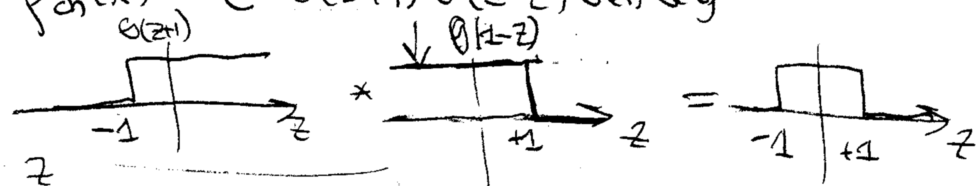
In Cartesian $\rho_{ch}(\vec{x}) = \rho_{ch}(x,y,z) = \underbrace{C}_{\text{constant}}\,\delta(x)\,\delta(y)$

What if it only went from $0 \le z < \infty$?

$\Rightarrow \rho_{ch}(\vec{x}) = C\,\theta(z)\,\delta(x)\,\delta(y)$ ← step or theta function

What if from $-1 \le z \le 1$?

$\Rightarrow \rho_{ch}(\vec{x}) = C\,\theta(z+1)\,\theta(1-z)\,\delta(x)\,\delta(y)$ ← or $\theta(z+1)-\theta(z-1)$



Recall $\theta(z) = \int_{-\infty}^{z} \delta(u)\,du = \begin{cases} 1 & \text{if } z>0 \\ 0 & \text{if } z<0 \end{cases}$ and $\boxed{\dfrac{d\theta(z)}{dz} = \delta(z)}$

Find C by the condition that length L has charge $\lambda \cdot L$

$\Rightarrow$ charge $\lambda L = \int_0^L dz \int_{-\infty}^{\infty} dx \int_{-\infty}^{\infty} dy\, \rho(\vec{x}) = CL \Rightarrow \boxed{C = \lambda}$

units: $\rho = 1/(\text{length})^3$ so $C \propto 1/\text{length}$ ✓

10/31/11

What about other coordinates?

What is $\delta(\vec{x}-\vec{x}_0)$ in spherical and cylindrical?

Spot the Error!

$$\delta(\vec{x}-\vec{x}_0) \stackrel{?}{=} \delta(r-r_0)\,\delta(\theta-\theta_0)\,\delta(\phi-\phi_0) \qquad \vec{x}=(r,\theta,\phi)$$

$$\stackrel{?}{=} \delta(\rho-\rho_0)\,\delta(\phi-\phi_0)\,\delta(z-z_0) \qquad \vec{x}=(\rho,\phi,z)$$

Doesn't work that $\int_{\text{all space!}} \delta(\vec{x}-\vec{x}_0)\, d^3x = 1$

because $d^3x \to r^2 \sin\theta \, d\theta \, d\phi \, dr$ spherical

or $d^3x \to \rho \, d\phi \, d\rho \, dz$

$$\Rightarrow \delta(\vec{x}-\vec{x}_0) = \frac{1}{r^2\sin\theta}\delta(r-r_0)\,\delta(\theta-\theta_0)\,\delta(\phi-\phi_0) \stackrel{\text{or}}{=} \frac{\delta(r-r_0)\,\delta(\cos\theta-\cos\theta_0)\,\delta(\phi-\phi_0)}{r^2}$$

$$\delta(\vec{x}-\vec{x}_0) = \frac{1}{\rho}\delta(\rho-\rho_0)\,\delta(\phi-\phi_0)\,\delta(z-z_0)$$

How do we deal with $\vec{x}_0 \to 0$?

Then θ_0 and ϕ_0 don't matter $\Rightarrow$ average over these is one way

$$\delta(\vec{x}) \to \frac{1}{4\pi r^2}\delta(r) \qquad \text{or} \qquad \delta(\vec{x}) \to \frac{1}{2\pi\rho}\delta(\rho)\,\delta(z)$$

requires $\int_0^\infty dr\, \delta(r) = 1$ (not $\frac{1}{2}$) because this is $\delta(r-0^+)$

Check them!

So in Problem 6.17

a) $\rho_{ch}(\vec{x}) = \lambda\, \delta(x)\,\delta(y) = \lambda\left(\frac{\delta(\rho)}{\rho}\right)\frac{1}{2\pi} = \frac{\lambda}{2\pi\rho}\delta(\rho)$ check it!

(assumes $\int_0^\infty \delta(\rho)\, d\rho = 1$ (not $1/2$))

b) $\rho_{ch}(\vec{x}) = ?$ Next time!