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$\boxed{\text{834 Lecture 12}}$

Lecture Plan

- finish up delta function discussion
- start on Fourier transforms

ever hopeful:
- solving S-eqn for bound states (Numerov method) 68 - 70

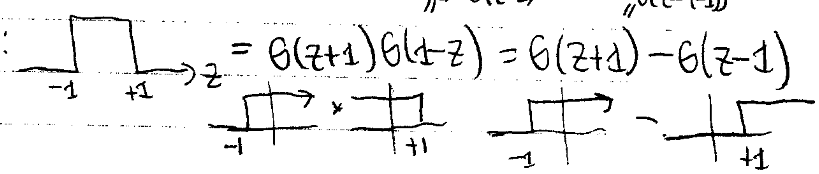
Before class:

- return rest of PS#5 up front (see me for grading issues)
- start up 834 page + Mathematica
- start up DOS Box and CUPS Waves and Optics $\to$ Fourier

On board: θ functions:

$$= \theta(z+1)\theta(1-z) = \theta(z+1) - \theta(z-1)$$

"$1-\theta(z-1)$" "$\theta(z-(-1))$"



Spot the error: (write correct expression)

Expand $f(t)$ in one $\underset{T}{\text{period}}$: $f(t) = \sum_{n=-\infty}^{\infty} c_n e^{in\pi t/T}$, $c_m = \frac{1}{2\pi}\int f(t)\, e^{im\pi t/T}\, dt$

Correct expressions: $f(t) = \sum_{n=-\infty}^{\infty} c_n e^{in\,2\pi t/T}$, $c_m = \frac{1}{T}\int_0^T f(t)\, e^{-im\,2\pi t/T}\, dt$

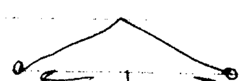
(class fill in)

$\leftarrow \frac{2}{T}$ or $\frac{1}{T}$ or $\frac{1}{2\pi}$? check!

Analysis: · How do you know what is up in the exponent?

ans: For $n=1$, the exponent should go from 0 to $2\pi i$ $\Rightarrow$ one full period.

- But what about those expansions for strings



where $y(x,0) = \sum_{n=1}^{\infty} a_n \sin\left(\frac{n\pi x}{L}\right)$, $a_n = \frac{2}{L}\int_0^L y(x,0)\sin\frac{n\pi x}{L}\,dx$??

Really

$n=1$ only use $x>0$

ans: The period here is not L, but $2L \Rightarrow -L \le x \le L$ (look at $n=1$ term!)

$\Rightarrow y(x,0) = \sum_{n=1}^{\infty} a_n \sin\left(n\frac{2\pi x}{(2L)}\right)$ ✓ Always check a term!

- The prefactor is easy: just set $n=m$ $\frac{1}{T}\int_0^T c_n\, dt = c_n$. Couldn't be $\frac{1}{2\pi}$ by units!
- Again, plug in $f(t)$ to see how c_m is projected, this tells you $-$ in exponent.

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Start with page 109 and $\delta(\vec{x}-\vec{x}_0)$ in spherical and cylindrical, getting to $\vec{x}_0 \to 0$ limit.

see Hassani, mathematical Physics, pg. 593 for more rigorous treatment

· If we take the actual limit with $\vec{x}_0 = (r_0, \theta_0, \phi_0)$ in spherical coordinates, then we could keep θ_0 and ϕ_0 fixed and take $r_0 \to 0^+$.

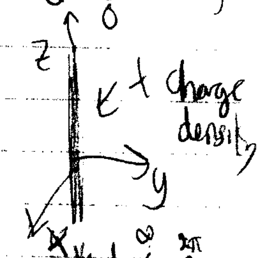
$$\Rightarrow \delta(\vec{x}-\vec{x}_0) \longrightarrow \frac{1}{r^2}\,\delta(r)\,\delta(\cos\theta - \cos\theta_0)\,\delta(\phi-\phi_0)$$

Does the result for $\int d^3x\, f(\vec{x})\,\delta(\vec{x})$ depend on θ_0 and ϕ_0?

· No, if $f(\vec{x})$ is continuous (uniform limit as $\vec{x}\to 0$, so same from any direction)

· So averaging over angles is just as good: $\delta(\vec{x}) = \frac{1}{4\pi r^2}\delta(r)$

Because $r_0 \to 0^+$, $\int_0^\infty \delta(r-0^+)\,dr \equiv \int_0^\infty \delta(r)\,dr = 1$ (and not $\frac{1}{2} = \int_0^\infty \delta(x)\,dx$)



So back to problem 6.17, where $\rho_{ch}(\vec{x}) = \lambda\,\delta(x)\,\delta(y)$

· Take the limit in cylindrical and spherical coordinates

① find delta functions in coordinates that take on single values.

② Then determine function that multiplies by integrating over appropriate region.

· Cylindrical: $\lambda \frac{1}{\rho}\delta(\rho-\rho_0)\,\delta(\phi-\phi_0)$ with $\delta(\rho-\rho_0) \to \delta(\rho-0^+)$, $\delta(\phi-\phi_0) \to \frac{1}{2\pi}$ $\longrightarrow \rho_{ch}(\vec{x}) = \frac{\lambda}{2\pi\rho}\delta(\rho) \Rightarrow \int_0^L dz \int_0^\infty \rho\, d\rho \int_0^{2\pi} d\phi\, \rho_{ch}(\vec{x}) = \lambda L$ ✓

where $\int_0^\infty \delta(\rho)\,d\rho = 1$

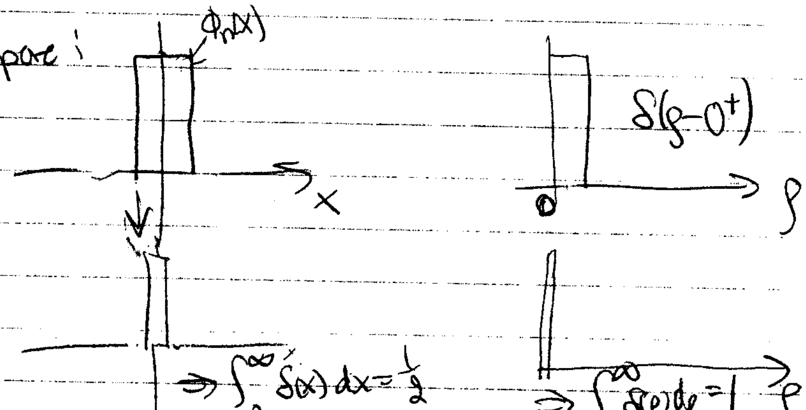
note: this is sometimes taken to be $\frac{1}{2}$!

· Spherical: $\lambda \frac{1}{2\pi r^2 \sin\theta}\left(\delta(\theta) + \delta(\theta-\pi)\right)$

$\int_0^L r^2 dr \int_0^{\pi} \sin\theta\, d\theta \int_0^{2\pi} d\phi\; \rho(\vec{x}) = 2\lambda L$ ✓

both directions ($+z$ and $-z$)

Compare:



$\Rightarrow \int_0^\infty \delta(x)\,dx = \frac{1}{2}$ $\quad\quad$ $\Rightarrow \int_0^\infty \delta(\rho)\,d\rho = 1$

This may be considered a mnemonic rather than a formal analysis!

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Important equations we'll see again:

$$\vec{\nabla} \cdot \frac{\hat{r}}{r^2} = +4\pi\delta(\vec{r})$$

or $\nabla^2\left(\frac{1}{r}\right) = -4\pi\delta(\vec{r})$ (note $\vec{\nabla}\frac{1}{r} = -\frac{\hat{r}}{r^2}$)

On homework, $\nabla_2^2\left(\ln\frac{\rho}{a}\right) = 2\pi\delta(\vec{\rho})$ in the xy plane

where $\nabla_2^2 \equiv \frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2}$, a is any constant, $\rho = \sqrt{x^2+y^2}$

Prove these by usual steps:

i) Easy to show $\nabla^2\left(\frac{1}{r}\right) = 0$ for $r \neq 0$ (we've done this!)

ii) show sifting property

$$\int \nabla^2\left(\frac{1}{r}\right) f(\vec{r})\, d^3r = -4\pi f(0)$$

Proofs in Leo, Boas, etc. ⇒ use divergence and other vector calculus theorems, but it's non-trivial ⇒ some terms cancel

Extension:

$$\boxed{\nabla^2 \frac{1}{|\vec{r}-\vec{r}_0|} = -4\pi\delta(\vec{r}-\vec{r}_0)}$$

we'll return to this!

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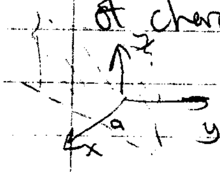
An E/M example (Lea example 6.5)

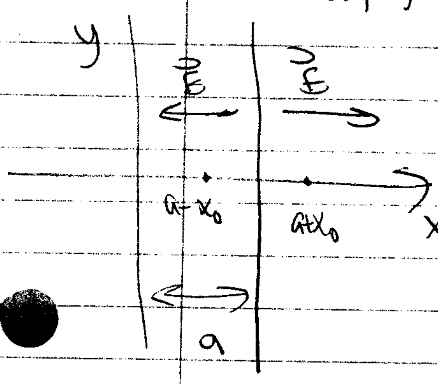
- Find the solution for the electric field due to a sheet of charge with charge density $\rho(\vec{x}) = \sigma_0 \delta(x-a)$



units $\frac{\text{charge}}{\text{volume}}$ units $\frac{\text{charge}}{\text{area}}$ units $\frac{1}{\text{length}}$

in general depends on x, y, z

$$\vec{E} = (E_x, 0, 0) \Rightarrow \frac{dE_x(x)}{dx} = \frac{\sigma_0}{\varepsilon_0}\delta(x-a) \quad \left(\text{from } \vec{\nabla}\cdot\vec{E} = \frac{\rho}{\varepsilon_0}\right)$$



Usually would invoke a gaussian cylinder!

Instead, integrate from $a-x_0$ to $a+x_0$

$$\int_{a-x_0}^{a+x_0} \frac{dE_x}{dx}\cdot dx = E_x(a+x_0) - E_x(a-x_0) = \int_{a-x_0}^{a+x_0} \frac{\sigma_0}{\varepsilon_0}\delta(x-a)\,dx = \frac{\sigma_0}{\varepsilon_0}$$

By symmetry, $E_x(a+x_0) = -E_x(a-x_0)$ [can we get this any other way?]

$$\Rightarrow 2E_x(a+x_0) = \frac{\sigma_0}{\varepsilon_0} \quad \text{or} \quad E_x(a+x_0) = \frac{\sigma_0}{2\varepsilon_0}$$

$$\Rightarrow \boxed{E_x(x) = \frac{\sigma_0}{2\varepsilon_0}\left(\theta(x-a) - \theta(a-x)\right)}$$

Plug in to check the equation

$$\frac{dE_x}{dx} = \frac{\sigma_0}{2\varepsilon_0}\left(\frac{d\theta(x-a)}{dx} - \frac{d\theta(a-x)}{dx}\right)$$

$\frac{d}{dx}\theta(x-a) = \delta(x-a)$

$\frac{d}{dx}\theta(a-x) = -\delta(x-a)$

$$= \frac{\sigma}{2\varepsilon_0}\left(\delta(x-a) - (-\delta(x-a))\right)$$

$$= \frac{\sigma}{\varepsilon_0}\delta(x-a) \checkmark$$

important to get the signs!

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Fourier Transforms: Core Competencies

① definition and basic Fourier transforms ← completing the square ← Gaussian, exponential, delta funct.
② Using Mathematica for complicated transforms
③ Use Fourier transforms to solve partial differential equations

Definition

$$F(k) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} f(x)\, e^{-ikx}\, dx \qquad \text{transform}$$

$$f(x) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} F(k)\, e^{ikx}\, dk \qquad \text{inverse}$$

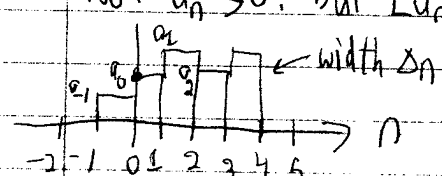
but symmetric (with − sign in exponent)

conventions:
- notation for $F(k)$: sometimes $\tilde{f}(k)$, sometimes just $f(k)$
- product of factors out front is $1/2\pi$ but different fields distribute differently (eg. $\int d^3x \Leftrightarrow \int \frac{d^3k}{(2\pi)^3}$)
- which exponential has the − sign (x,k) vs. (t,ω), for example

Comparison to Fourier Series: $f(x) = \sum_{n=-\infty}^{\infty} a_n e^{in\pi x/L}$ for $-L \le x \le +L$

Can we just take $L \to \infty$? $\qquad a_n = \frac{1}{2L} \int_{-L}^{L} f(x)\, e^{-in\pi x/L}\, dx$

No: $a_n \to 0$. But $L a_n$ would be ok.



$$\Rightarrow f(x) = \sum_{n=-\infty}^{\infty} a_n e^{in\pi x/L} \Delta n \underset{k=\frac{n\pi}{L}}{\overset{\text{rescale } \Delta n \text{ to}}{=}} \sum_n \left(\frac{L}{\pi} a_n\right) e^{i\left(\frac{n\pi}{L}\right)x} \frac{\Delta n\, \pi}{L}$$

sum is like an integral if we could take $\Delta n \to 0$.

$$\Rightarrow \Delta k = \Delta n \frac{\pi}{L} = \sum_{k(n)} \left(\frac{L}{\pi} a_n\right) e^{ikx} \Delta k \xrightarrow[L\to\infty]{} \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} F(k)\, e^{ikx}\, dk$$

where $\frac{1}{\sqrt{2\pi}} F(k) = \frac{L}{\pi} a\left(k = \frac{n\pi}{L}\right)$ [here $a(k)$ becomes continuous]

$$\Rightarrow F(k) = L\sqrt{\frac{2}{\pi}}\, a(k) \Rightarrow L\sqrt{\frac{2}{\pi}}\, a_n \xrightarrow[L\to\infty]{} F(k) = \sqrt{\frac{2}{\pi}} \frac{1}{2} \int_{-\infty}^{\infty} f(x)\, e^{-ikx}\, dx \quad \checkmark$$

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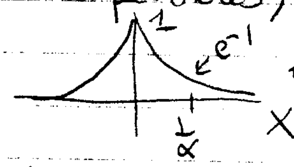
- Key examples: check with Mathematica FourierTransform

- Note: You should be able to do the manipulations here, not just using Mathematica (core competencies)

(A) $f(x) = e^{-\alpha|x|}$ exponential



"width" $\Rightarrow$ standard deviation $\Delta x \sim 1/\alpha$

$$F(k) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} e^{-\alpha|x|} e^{-ikx}\, dx$$ ← just combine exponentials and integrate as usual

split so $|x|$ can be separated

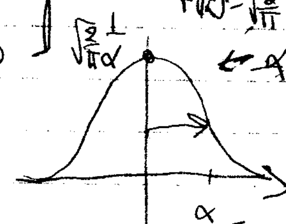
$$= \frac{1}{\sqrt{2\pi}} \left[\int_{-\infty}^{0} e^{(\alpha - ik)x}\, dx + \int_{0}^{\infty} e^{-(\alpha+ik)x}\, dx \right]$$

$x<0 \Rightarrow |x| = -x$ $\qquad$ $x>0 \Rightarrow |x| = +x$

$$= \frac{1}{\sqrt{2\pi}} \left[\frac{1}{\alpha - ik} e^{(\alpha-ik)x} \Big|_{-\infty}^{0} + \frac{e^{-(\alpha+ik)x}}{-(\alpha+ik)} \Big|_{0}^{\infty} \right]$$

$$= \frac{1}{\sqrt{2\pi}} \left(\frac{1}{\alpha - ik} - \left(\frac{-1}{\alpha+ik} \right) \right) = \boxed{\sqrt{\frac{2}{\pi}} \frac{\alpha}{\alpha^2 + k^2}}$$

For k small, $F(k) = \sqrt{\frac{2}{\pi}} \frac{1}{\alpha} \frac{1}{1 + k^2/\alpha^2}$ ← $\alpha\left(1 - \frac{k^2}{\alpha^2}\right)$



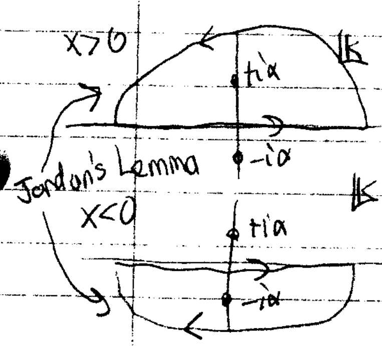
← $\Delta k \alpha\, \alpha$

- product of widths, $\Delta x \Delta k \sim \frac{1}{\alpha} \cdot \alpha \sim 1$! (up to numerical factors)

More precise: $\Delta x \equiv \sqrt{\langle x^2 \rangle - \langle x \rangle^2}$ (use normalized distribution for easiest calculation of $\langle x^2\rangle$, $\langle x \rangle$)

Now the inverse: This should look familiar (from the midterm!)

$$f(x) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} \sqrt{\frac{2}{\pi}} \frac{\alpha}{\alpha^2 + k^2} e^{ikx}\, dk = \frac{\alpha}{\pi} \int_{-\infty}^{\infty} \frac{e^{ikx}}{(k+i\alpha)(k-i\alpha)}$$



$$x>0: f(x) = \frac{\alpha}{\pi} 2\pi i \left(\frac{e^{-\alpha x}}{2i\alpha} \right) = e^{-\alpha x}$$

$$x<0: f(x) = \frac{\alpha}{\pi} (-2\pi i) \frac{e^{i(-i\alpha)x}}{-2i\alpha} = e^{\alpha x}$$

↖ clockwise

$\Big\}\ e^{\alpha|x|}$ ✓

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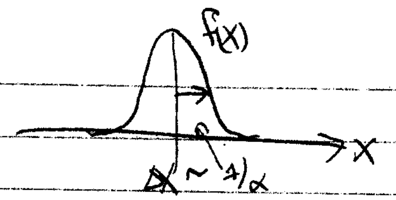
(B) Gaussian $f(x) = N e^{-\alpha^2 x^2}$



$$F(k) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} N e^{-\alpha^2 x^2} e^{-ikx} dx$$

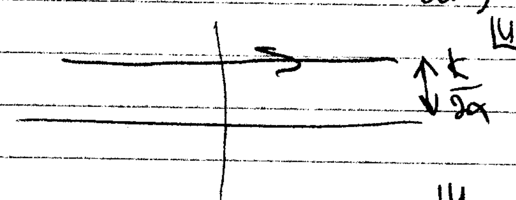
observe $e^{-\alpha^2 (x+\beta)^2} = e^{-\alpha^2 x^2} e^{-2\alpha^2 \beta x} e^{-\alpha^2 \beta^2}$

$$\Rightarrow \underbrace{e^{-\alpha^2 x^2 - 2\alpha^2 \beta x}}_{\text{simplify terms}} = e^{-\alpha^2 (x+\beta)^2} e^{+\alpha^2 \beta^2} \Rightarrow 2\alpha^2 \beta = ik \Rightarrow \beta = \frac{ik}{2\alpha^2}$$

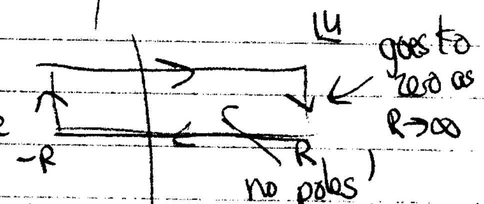
$$\alpha^2 \beta^2 = \alpha^2 \left(-k^2/4\alpha^4\right)$$

$$\Rightarrow F(k) = \frac{1}{\sqrt{2\pi}} e^{-\frac{k^2}{4\alpha^2}} N \int_{-\infty}^{\infty} e^{-\alpha^2 \left(x + \frac{ik}{2\alpha^2}\right)^2} dx$$ change to $u = \alpha\left(x + \frac{ik}{2\alpha^2}\right)$

$$= \frac{N}{\sqrt{2\pi}} e^{-\frac{k^2}{4\alpha^2}} \int_{-\infty + \frac{ik}{2\alpha}}^{\infty + \frac{ik}{2\alpha}} e^{-u^2} \frac{du}{\alpha}$$



$$= \frac{N}{\sqrt{2\pi}} e^{-\frac{k^2}{4\alpha^2}} \frac{1}{\alpha} \underbrace{\int_{-\infty}^{\infty} e^{-u^2} du}_{= \sqrt{\pi}}$$ because



$\Rightarrow$ integral on top = integral on bottom

$$\boxed{F(k) = \frac{N}{\alpha\sqrt{2}} e^{-\frac{k^2}{4\alpha^2}}}$$

Gaussian $\xrightarrow[\text{transform}]{\text{Fourier}}$ Gaussian

width $\Delta k \sim 2\alpha$ so $\Delta x \Delta k \sim \frac{1}{\alpha} 2\alpha \sim 1$ again

[More precise with $\Delta x = \sqrt{\langle x^2 \rangle - \langle x \rangle^2}$

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© $f(x) = 1 \Rightarrow$ delta function

$$F(k) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} 1 \cdot e^{-ikx} dx = \sqrt{2\pi}\, \delta(k)$$

$$\text{or} \quad \delta(k) = \frac{1}{2\pi} \int_{-\infty}^{\infty} e^{-ikx} dx \qquad \delta(x) = \frac{1}{2\pi} \int_{-\infty}^{\infty} e^{+ikx} dk$$

Properties of Fourier transform F[ ]

- can you prove these?

- $F[f+g] = F[f] + F[g] \Leftarrow \int (f(x)+g(x))\, e^{-ikx} dx = \int f(x) e^{-ikx} dx + \int g(x) e^{-ikx} dx$
- $F[af] = aF[f] \quad \Leftarrow \int a f(x)\, e^{-ikx} dx = a \int f(x)\, e^{-ikx} dx$
- $F\left[\frac{df}{dx}\right] = ikF(k)$
- $F^*(k) = F(-k)$

Parseval: $\displaystyle\int_{-\infty}^{\infty} |f(x)|^2 dx = \int_{-\infty}^{\infty} |F(k)|^2 dk$

so normalized distribution in x is also normalized in k!

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Application to a PDE (partial differential equation): Diffusion

• physical situation: at $t=0$, total mass m at $x=\ell$ in long pipe of cross section A, full of water. Find $t>0$ distribution

Or, temperature of rod initially at T_0, heated at one end to T_1 at $t=0$.

equation: $$\frac{\partial \rho(x,t)}{\partial t} = \underset{\uparrow \text{ diffusion constant}}{D} \frac{\partial^2 \rho(x,t)}{\partial x^2}$$ (cf. time-dependent Schrödinger equation)

ρ is mass distribution (integrate over x to get m)

$\rho(x, t<0) = 0$ and assume $\rho(x,t) \xrightarrow[x\to\infty]{} 0$ [physically reasonable]

Initial condition: $\rho(x,0) = \frac{m}{A}\delta(x-L)$ [check: $\int_{\text{pipe}} d^3x\, \rho(x,0) = \frac{m}{A}\cdot A = m$ ✓]

Let $\tilde{\rho}(k,t) = \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} \rho(x,t)\, e^{-ikx}\, dx$

Plan: "apply" $\frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} e^{-ikx} dx\, [\ ]$ to both sides of diffusion equation

On left, pull $\frac{\partial}{\partial t}$ from integral $\Rightarrow \frac{\partial \tilde{\rho}(k,t)}{\partial t}$ (still a partial derivative!)

On right, $\frac{D}{\sqrt{2\pi}}\int_{-\infty}^{\infty} e^{-ikx} \frac{\partial^2 \rho(x,t)}{\partial x^2} dx \Rightarrow$ partially integrate twice (no surface terms because $\rho(\pm\infty)=0$)

$$= \frac{D}{\sqrt{2\pi}}(-ik)^2 \int_{-\infty}^{\infty} e^{-ikx} \rho(x,t)\, dx = -k^2 D\,\tilde{\rho}(k,t)\ !$$

$$\Rightarrow \boxed{\frac{\partial \tilde{\rho}(k,t)}{\partial t} = -k^2 D \tilde{\rho}(k,t)}$$ but this is simple for any given value of k!

Solve by inspection: $\tilde{\rho}(k,t) = \tilde{\rho}_0(k)\, e^{-k^2 D t}$ and find $\tilde{\rho}_0$ from initial condition.

$$\tilde{\rho}_0(k) = \tilde{\rho}(k,0) = \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} \frac{m}{A}\delta(x-L)\, e^{-ikx} dx = \frac{1}{\sqrt{2\pi}}\frac{m}{A} e^{-ik\ell} \Rightarrow \boxed{\tilde{\rho}(k,t) = \frac{m}{A}\frac{e^{-ik\ell}}{\sqrt{2\pi}} e^{-k^2 D t}}$$

Find $$\rho(x,t) = \frac{m}{A}\frac{1}{2\pi}\int_{-\infty}^{\infty} e^{(ikx - ik\ell - k^2 D t)} dk = \frac{m}{A}\frac{1}{2\pi}\int_{-\infty}^{\infty} e^{-\left(\sqrt{Dt}\,k + i\frac{(\ell-x)}{2\sqrt{Dt}}\right)^2} e^{-\frac{1}{4}\frac{(x-\ell)^2}{Dt}} = \boxed{\frac{m}{A}\frac{1}{2\sqrt{\pi D t}}\, e^{-\frac{(x-\ell)^2}{4Dt}}}$$

complete square!

note $1/\sqrt{t}$

← Gaussian spread