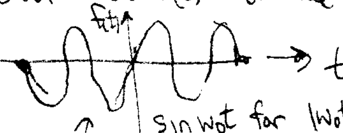
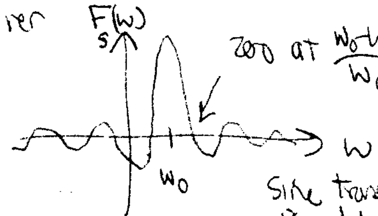
on board ⇒ Warm-up/recap: What will the Fourier transform of f(t) look like? (sine)



⇒



zero at $\frac{w_0 - w}{w_0} = \pm\frac{1}{3}$ ← $\pm\frac{1}{N}$ to first minimum

(136)

Sine transform because f(t) is odd

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$\boxed{\text{834 Lecture 15}}$

$w_0 \Delta t \doteq 2\pi N$ ← N = # of cycles

spread in w → $\frac{\Delta w_0}{w_0} \doteq \frac{1}{N}$ ⇒ $\Delta w_0 \Delta t = \frac{1}{N} 2\pi N = 2\pi$

Lecture Plan:

- Overview of Sturm-Liouville theory
- Converting non-self-adjoint operators to self-adjoint
- Recursion (recurrence) relations
- Follow-up (if time) on undetermined coefficients (130)

Before class:

- Start up 834 page + Mathematica
  - legendre_polynomial_1.nb
  - laguerre_polynomial_1.nb
- bring up landau_conventions_p.pdf ← start with this to motivate where we're going

On board: • PS#7: If a problem says to use Mathematica to check, that means you must do so!
⇒ Mathematica use is part of the course

General Sturm-Liouville form for y(x)

$$\frac{d}{dx}\left(f(x)\frac{dy}{dx}\right) - g(x)\, y + \lambda w(x)\, y = 0$$

Given: $w(x) \geq 0$ on $a \leq x \leq b$, $f(x)$, $g(x)$

Physics dictates b.c.'s: $\left.\left(\alpha_1 y + \beta_1 \frac{dy}{dx}\right)\right|_{x=a} = 0$ $\left.\left(\alpha_2 y + \beta_2 \frac{dy}{dx}\right)\right|_{x=b} = 0$

Special cases: $\alpha = 0 \Rightarrow \frac{dy}{dx} = 0$ "Neumann"

$\beta = 0 \Rightarrow y = 0$ "Dirichlet"

Goal: Find λ's (eigenvalues) for which there are non-trivial (i.e. $y(x) \neq 0$) eigenfunctions $y_n(x)$.

Properties: ① orthogonal wrt w(x) $\int_a^b w(x) y_m(x) y_n(x)\, dx = C_m \delta_{mn}$ ($C_m = 1$ if normalized)

② Completeness on $x \in [a,b]$: $f(x) = \sum_{n=0}^{\infty} a_n y_n(x)$, $a_m = \frac{1}{C_m}\int_a^b f(x) y_m(x) w(x)\, dx$

$\delta(x - x') = \sum_n y_n(x) y_n(x') w(x') \, 1/C_n$

③ real eigenvalues

④ self-adjoint: $\int_a^b y_m \mathcal{L} y_n \, dx = \int_a^b y_n (\mathcal{L} y_m)\, dx$ with $\mathcal{L} y \equiv \frac{d}{dx}\left(f(x)\frac{dy}{dx}\right) - g(x) y$

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Basic examples:

Helmholtz: $\frac{d^2y}{dx^2} + k^2 y = 0$ like in PS#8

General solution depends on $k^2 > 0$ or $k^2 < 0$

$k^2 > 0$, $y_n = C_1^{(n)} \cos k_n x + C_2^{(n)} \sin k_n x$ or $D_1^{(n)} e^{ik_n x} + D_2^{(n)} e^{-ik_n x}$

e.g., Boundary conditions: $ay + by'|_{x=0} = 0$; $\alpha y + \beta y'|_{x=L} = 0$

- Linear and Homogeneous eq + BC's $\Rightarrow$ if y_n is a solution, then so is $c y_n$ for any c.
  $\Rightarrow$ only ratio $C_2^{(n)}/C_1^{(n)}$ is determined.
- a, b, α, β are given by the <u>physics</u>
- $\Rightarrow$ 2 conditions but only one constant $\Rightarrow$ only certain k_n^2's
  $\Rightarrow$ eigenvalue problem. But usually not closed form $\Rightarrow$ transcendental equation for k_n
  - Example in PS#8

- All the properties on the summary (on 136)

- Run through the Laplace's equation example in cylindrical from 133

- Then look at some general properties, starting with 135

- Then go back to another example: Legendre polynomials

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If we consider $\nabla^2\Phi = 0$ in spherical coordinates, separating the θ dependence from r and φ leads to

$$\frac{1}{\sin\theta}\frac{d}{d\theta}\left(\sin\theta\frac{dP}{d\theta}\right)\frac{1}{P} - \frac{m^2}{\sin^2\theta} + k = 0 \quad \text{for } P(\theta)$$

Then $x = \cos\theta$, $dx = -\sin\theta\, d\theta$ gives us

$$\frac{d}{dx}\left((1-x^2)\frac{dP}{dx}\right) - \frac{m}{1-x^2} + kP(x) = 0$$

associated Legendre Functions

Check against general form $\Rightarrow$ $\begin{cases} f(x) = 1-x^2 \\ g(x) = 0 \\ w(x) = 1 \end{cases}$

Where do Legendre polynomials come from?

If no dependence on φ, $m=0$ and look for power series solution;

Frobenius $y = \sum_{n=0}^{\infty} a_n x^n$

usual procedure $\Rightarrow$ $a_{p+2} = a_p \frac{p(p+1)-k}{(p+2)(p+1)}$ $\Rightarrow$ terminates for $k = l(l+1)$ for integer l.

If not terminated, then converges for $-1<x<+1$ but diverges for $x=\pm 1$, where we want a solution (eg. for Φ). $\Rightarrow$ require $k = l(l+1)$

$$\Rightarrow \frac{d}{dx}\left((1-x^2)\frac{dP_l}{dx}\right) + l(l+1)P_l = 0$$ and choose $P_l(1) \equiv 1$ for all l.

$$\Rightarrow \int_{-1}^{1} P_l(x)P_{l'}(x)\,dx = \delta_{ll'}\frac{2}{2l+1}$$

see legendre_polynomial_1.nb for examples.

l	$P_l(x)$
0	1
1	x
2	$\frac{1}{2}(3x^2-1)$
3	$\frac{x}{2}(5x^2-3)$

The P_l satisfy various recursion relations.

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Where do recursion relations for orthogonal polynomials come from?

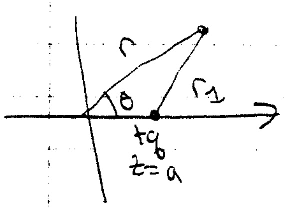
- One source is a generating function that can be associated with a set of polynomials.
  - This is a function whose expansion has the desired polynomials as coefficients.

- For Legendre polynomials

$$g(t,x) = \frac{1}{\sqrt{1-2xt+t^2}} = \sum_{n=0}^{\infty} P_n(x)\, t^n$$

[This is Arken's notation, in Lea $g(t,x) \Rightarrow G(x,\mu)$].

- Comes from solution to Laplace's equation $\nabla^2 \Phi = 0$



$$\Phi = \frac{q}{4\pi\varepsilon_0 r_1} = \frac{q}{4\pi\varepsilon_0} \frac{1}{(r^2+a^2-2ar\cos\theta)^{1/2}}$$

$$= \frac{q}{4\pi\varepsilon_0 r} \frac{1}{1-2ar\cos\theta+a^2} = \frac{q}{4\pi\varepsilon_0 r} \sum_{n=0}^{\infty} P_n(\underset{x}{\cos\theta}) \underset{t}{\left(\frac{a}{r}\right)}^n$$

More on this later!

- For now, note that $g(t,1) = \frac{1}{1-t} = \sum_{n=0} t^n \Rightarrow P_n(1) = 1$ matching coefficients
  - similarly $g(t,-1) = 1/1+t \Rightarrow P_n(-1) = (-1)^n$
  - Others: $P_n(-x) = (-1)^n P_n(x)$ [parity] and $|P_n(\cos\theta)| \le P_n(1) = 1$ ← clear from plots

- Now $\frac{\partial g}{\partial t}(t,x) = \frac{x-t}{(1-2xt+t^2)^{3/2}} = \sum_{n=0}^{\infty} n P_n(x) t^{n-1}$

$$= \frac{x-t}{1-2xt+t^2} \cdot \frac{1}{(1-2xt+t^2)^{1/2}} = \frac{x-t}{1-2xt+t^2} \sum_{n=0}^{\infty} t^n P_n(x)$$

$$\Rightarrow (1-2xt+t^2)\sum_{n=0}^{\infty} n P_n(x) t^{n-1} + (t-x)\sum_{n=0} P_n(x) t^n = 0 \Rightarrow \text{equate coefficients of } t^m$$

$$\Rightarrow (2n+1)\, x P_n(x) = (n+1) P_{n+1}(x) + n P_{n-1}(x), \quad n = 1, 2, 3, \ldots$$

- Differentiate wrt x to get others, Look these up as needed!

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Example of using recursion relations: Lea problem 8.5.

Show using recursion relations (and integration by parts) that

$$\int_{-1}^{1} P'_{l_1}(x)\, P'_{l_2}(x)\,(1-x^2)\, dx = 0 \text{ if } l_1 \neq l_2$$

- follows as 3(d) in PS#8 by different method.
- Assume we know

$$\int_{-1}^{1} P_l(x)\, P_{l'}(x)\, dx = \frac{2}{2l+1}\delta_{ll'} \quad \text{and} \quad \overset{\text{for surface terms}}{\downarrow} P_l(1)=1,\ P_l(-1)=(-1)^l$$

and various recursion relations:

ⓐ $l P_{l-1}(x) - (2l+1)x P_l(x) + (l+1)P_{l+1}(x) = 0$
ⓑ $P_l(x) = P'_{l-1}(x) - 2x P'_l(x) + P'_{l+1}(x)$
ⓒ $l P_l(x) = x P'_l(x) - P'_{l-1}(x)$
ⓓ $P_{l-1}(x) = x P_l(x) + (1-x^2)P'_l(x)/l$

How do we proceed? only one of these has a $(1-x^2)$ factor, so use that first

ⓓ $\Rightarrow (1-x^2)P'_{l_2}(x) = -l_2[x P_{l_2}(x) - P_{l_2-1}(x)]$

$$\Rightarrow = (-l_2)\int_{-1}^{1} \frac{dP_{l_1}}{dx}[x P_{l_2} - P_{l_2-1}]dx$$

. Now we note that ⓒ has $xP'_l - P'_{l-1}$ $\Rightarrow$ integrate by parts:

$$= -l_2\left[P_{l_1}(x P_{l_2} - P_{l_2-1})\right]\Big|_{-1}^{1} + l_2\int_{-1}^{1} P_{l_1}\cdot[P_{l_2} + \underbrace{x P'_{l_2} - P'_{l_2-1}}_{l_2 P_{l_2}}]\,dx$$

$$-l_2(1\cdot(1-1) - 1\cdot((-1)^{l_2+1} - (-1)^{l_2-1}))$$ $= 0$

$$\Rightarrow \quad = l_2(l_2+1)\int_{-1}^{1} P_{l_1} P_{l_2}\, dx = \frac{2l_1(l_1+1)}{2l_1+1}\delta_{l_1 l_2} \quad \text{QED}$$

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What if an equation is not in the form of the S-L equation?

Example: Laguerre $xy'' + (1-x)y' + ny = 0$ on $x \in [0, \infty]$

The differential operator $x\frac{d^2}{dx^2} + (1-x)\frac{d}{dx}$ is not self-adjoint.

assume surface terms vanish and try:

$$\int y_1 \left(x\frac{d^2}{dx^2}\right) y_2 = -\int (y_1' x + y_1)\frac{d}{dx}y_2\,dx = +\int (y_1'' x + 2y_1')\, y_2\, dx$$

$$\int y_1 (1-x)\frac{d}{dx} y_2 = -\int \left(y_1'(1-x) - y_1\right) y_2\, dx$$

$$\Rightarrow \int y_1 \left(x\frac{d^2}{dx^2} + (1-x)\frac{d}{dx}\right) y_2\, dx = \int \left[\left(x\frac{d^2}{dx^2} + (1+x)\frac{d}{dx}\right) y_1\right] y_2\, dx$$

not equal

What can we do? The solutions are unchanged if we multiply by a positive function $p(x)$ [that is, $p(x) > 0$ for all x]

Equate $\frac{d}{dx} f(x) \frac{dy}{dx} - g(x) y + \lambda w(x) y = f y'' + \frac{df}{dx} y' - gy + \lambda w y$
$= p(x)\left[x y'' + (1-x) y'\right] + p(x) n y$

$\Rightarrow w(x) = p(x)$, $\lambda = n$, $f(x) = x\,p(x)$, $\frac{df}{dx} = (1-x)p(x)$
$g(x) \equiv 0$ $\quad \Rightarrow \frac{df}{dx} = p + xp' = (1-x)p$ or $p' = -p$.

Solving, $p(x) = e^{-x}$ (the overall magnitude doesn't matter)

$$\Rightarrow \boxed{\frac{d}{dx}\left(x e^{-x} \frac{dy}{dx}\right) + n e^{-x} y(x) = 0}$$

Laguerre polynomials! See laguerre_polynomial_1.nb for tests.

Laguerre polynomials show up in solutions to hydrogen atom, in 3D quantum mechanics (cf. Hermite polynomials in solutions to 1D harmonic oscillator)

11/14/11.

Consider solving a Laplace equation problem with spherical symmetry: Lea example 8.2: Hollow sphere conductor except for insulating strip along equator ($\theta = \frac{\pi}{2}$). Radius is a. Bottom half is grounded ($V=0$) while top half is held at $V=V_0$.
- What is the potential inside the sphere?

Plan: No charges inside, so solve $\nabla^2\Phi = 0$ subject to the boundary conditions of V_0 on the upper hemisphere and 0 on the lower hemisphere.

Separate variables in spherical coordinates: $\Phi = R(r)P(\theta)W(\phi)$ and (Jackson covers!):

$$\nabla^2\Phi = \frac{1}{r^2}\frac{\partial}{\partial r}\left(r^2\frac{\partial\Phi}{\partial r}\right) + \frac{1}{r^2\sin\theta}\frac{\partial}{\partial\theta}\left(\sin\theta\frac{\partial\Phi}{\partial\theta}\right) + \frac{1}{r^2\sin^2\theta}\frac{\partial^2\Phi}{\partial\phi^2} = 0$$

Usual game of substituting and dividing by Φ yields

$$\frac{1}{Rr^2}\frac{\partial}{\partial r}\left(r^2\frac{\partial R}{\partial r}\right) + \frac{1}{r^2\sin\theta}\frac{\partial}{\partial\theta}\left(\sin\theta\frac{\partial P}{\partial\theta}\right)\frac{1}{P} + \frac{1}{Wr^2\sin^2\theta}\frac{\partial^2 W}{\partial\phi^2} = 0$$

isolate simplest term: $W \Rightarrow$ multiply by $r^2\sin^2\theta$

$$\underbrace{\frac{\sin^2\theta}{R}\frac{\partial}{\partial r}\left(r^2\frac{\partial R}{\partial r}\right) + \sin\theta\frac{\partial}{\partial\theta}\left(\sin\theta\frac{\partial P}{\partial\theta}\right)\frac{1}{P}}_{\text{only } r,\theta} + \underbrace{\frac{1}{W}\frac{\partial^2 W}{\partial\phi^2}}_{\text{only } \phi} = 0$$

- need W to be periodic in ϕ with period $2\pi \Rightarrow \frac{1}{W}\frac{\partial^2 W}{\partial\phi^2} = -m^2 \Rightarrow W = e^{\pm im\phi}$ (or sin/cos $m\phi$)
(If $+m^2$, then not periodic).

- Then isolate first term by dividing by $\sin^2\theta$, to get

$$\underbrace{\frac{1}{R}\frac{\partial}{\partial r}r^2\frac{\partial R}{\partial r}}_{=k} + \underbrace{\frac{1}{\sin\theta}\frac{\partial}{\partial\theta}\left(\sin\theta\frac{\partial P}{\partial\theta}\right)\frac{1}{P} - \frac{m^2}{\sin^2\theta}}_{\text{only } \theta} = 0$$

$\leftarrow$ only r

The θ equation gives us again the associated Legendre equation we've already considered

$x = \cos\theta \Rightarrow \frac{d}{dx}\left((1-x^2)\frac{dP}{dx}\right) - \frac{m^2}{1-x^2}P + kP = 0.$

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What about the $R(r)$ dependence when $m=0$ (independent of ϕ) and $k=\ell(\ell+1)$.

$$\Rightarrow \frac{d}{dr}\left(r^2\frac{dR}{dr}\right) = \ell(\ell+1)R$$

[we'll use $\frac{d}{dr}$ rather than $\frac{\partial}{\partial r}$ since only r dependence]

If we let $R(r) = \frac{u(r)}{r}$

$$\Rightarrow \frac{d}{dr}\left(r^2\frac{dR}{dr}\right) = r\frac{d^2u}{dr^2} \Rightarrow \frac{d^2u}{dr^2} - \frac{\ell(\ell+1)u(r)}{r^2} = 0$$

(cf. radial 3D S-eqn with no potential at zero energy)

Solutions are powerlaws $\Rightarrow u=r^p$

$$\Rightarrow p(p-1)r^{p-2} - \ell(\ell+1)r^{p-2} = 0 \Rightarrow p(p-1)=\ell(\ell+1)$$

so $p=\ell+1$ or $p=-\ell$

$$\Rightarrow R(r) = r^\ell \text{ or } 1/r^{\ell+1}$$

This means that the general eigenfunction expansion for axisymmetry (no ϕ dependence) is

$$\boxed{\Phi(r,\theta) = \sum_{\ell=0}^{\infty}\left[A_\ell r^\ell + B_\ell r^{-(\ell+1)}\right]P_\ell(\cos\theta)}$$

and A_ℓ, B_ℓ are fixed by the boundary conditions in r.

So back to our 2 hemisphere problem.

i) There's nothing at $r=0$, so Φ can't diverge there $\Rightarrow B_\ell=0\ \forall \ell$.

ii) evaluate $\Phi(r,\theta)$ at $r=a$

$$\Rightarrow \sum_{\ell=0}^{\infty} A_\ell a^\ell P_\ell(x) = \begin{cases} V_0 & \text{if } 1\geq x>0 \quad (\cos\theta \text{ for } 0\leq\theta<\frac{\pi}{2}) \\ 0 & \text{if } 0>x\geq-1 \quad (\cos\theta \text{ for } \frac{\pi}{2}<\theta\leq\pi) \end{cases}$$

Use orthogonality to project out the A_ℓ's.

$$\Rightarrow \int_{-1}^{1}\sum_{\ell=0}^{\infty} A_\ell a^\ell P_\ell(x)P_{\ell'}(x)\,dx = \int_0^1 V_0 P_{\ell'}(x)\,dx \Rightarrow \frac{2}{2\ell+1}\delta_{\ell\ell'} \text{ on left side}$$

exchange $\ell'\to\ell \Rightarrow A_\ell a^\ell \frac{2}{2\ell+1}\, V\int_0^1 P_\ell(x)\,dx$

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How do we evaluate $\int_0^1 P_l(x)\,dx$?

For $l=0$, $\int_0^1 1 \cdot dx = 1$

For $l>0$, we can use the recursion relation $P_l(x) = \frac{1}{2l+1}\left(P'_{l+1}(x) - P'_{l-1}(x)\right)$ or the Rodrigues Formula:

$$P_l(x) = \frac{1}{2^l l!}\frac{d^l}{dx^l}(x^2-1)^l$$

or simply ask Mathematica!

$$\int_0^1 P_l(x)\,dx = \text{Integrate[LegendreP[l,x],\{x,0,1\}]}$$

$$= \frac{\sqrt{\pi}}{2}\frac{1}{\Gamma(1-\frac{l}{2})\Gamma(\frac{3+l}{2})}$$ which is 0 for even $l>0$. (This is clear since $\int_0^1 P_l(x)dx = \int_0^1 P_l(x)P_0(x)\,dx$ for even l, so δ_{l0}.)

$$\Rightarrow \Phi(r,\theta) = \frac{V_0}{2}\left[1 + \sum_{n\,\text{odd}} \frac{\sqrt{\pi}}{2}\frac{2n+1}{\Gamma(1-\frac{n}{2})\Gamma(\frac{3+n}{2})}\left(\frac{r}{a}\right)^l P_l(\cos\theta)\right]$$

- See legendre-polynomial-1.nb

- How would we solve this numerically?
- Is there a clear preference with the analytic solution given above?