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834 Lecture 2

Before class:

- have students sign sheet in order and take pictures. On board: "Please come have your picture taken. so I can learn names and faces."
- Log in to computer and set up with 834 page, hints, and Mathematica page. Start up Mathematica with some sample notebooks (including the one on 3D spherical integrals). Also load Jackson covers.
- Poll for Monday and Tuesday office hours

On the board:

- ε-δ warmups plus "Spot the Error!" (answers in lecture notes)

ε-δ

- $\vec{\nabla}\cdot(\vec{V}\times\vec{W}) = \vec{W}\cdot(\vec{\nabla}\times\vec{V}) - \vec{V}\cdot(\vec{\nabla}\times\vec{W})$ "product rule"
- $\vec{\nabla}\times(\vec{V}\times\vec{W}) = \vec{V}(\vec{\nabla}\cdot\vec{W}) - \vec{W}(\vec{\nabla}\cdot\vec{V}) + (\vec{W}\cdot\vec{\nabla})\vec{V} - (\vec{V}\cdot\vec{\nabla})\vec{W}$
- $\vec{\nabla}\times(\phi\vec{\nabla}\phi) = 0$

PS1.2: • $\vec{\nabla}\times\hat{\phi} \stackrel{?}{=}$ (cylindrical)

only place that A_2 survives

cylindrical: $A_1=0,\ A_2=1,\ A_3=0 \Rightarrow \hat{z}\frac{1}{\rho}\left(\frac{\partial}{\partial\rho}(\rho\cdot 1)\right) = \hat{z}\frac{1}{\rho}$

spherical: $A_1=0,\ A_2=0,\ A_3=1 \Rightarrow$ two terms!

- "Spot the Error!" (What is wrong in each of these?)

	problem
$(\vec{\nabla}\times\vec{V})\cdot(\vec{\nabla}\times\vec{W}) = \varepsilon_{abc}\frac{\partial}{\partial x_b}V_c\,\varepsilon_{def}\frac{\partial}{\partial x_e}W_f$	
$\vec{\nabla}\times(\vec{\nabla}\times\vec{V}) = \varepsilon_{abc}\frac{\partial}{\partial x_b}\varepsilon_{cde}\frac{\partial}{\partial x_d}V_e$	d should be a vector on left, a component on right
$(A\times B)_a + (C\times D)_d = \varepsilon_{abc}A_bB_c + \varepsilon_{def}C_eD_f$	d should be a
$\delta_{ab}\varepsilon_{acd}\varepsilon_{efa}\delta_{fd}$	a appears 3 times
$[A\times(B\times C)]_a = \varepsilon_{abc}A_b\varepsilon_{acd}B_cC_d$	c appears 3 times
$\vec{\nabla}\times(\vec{B}\cdot\vec{C}) = 0$	$\vec{B}\cdot\vec{C}$ is scalar

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Answers to warm-ups:

$$\left[\vec{\nabla}\times(\vec{V}\times\vec{W})\right]_a = \varepsilon_{abc}\frac{\partial}{\partial x_b}(\vec{V}\times\vec{W})_c = \varepsilon_{abc}\frac{\partial}{\partial x_b}\varepsilon_{cde}V_d W_e$$

$$= \underbrace{\varepsilon_{cab}\varepsilon_{cde}}\frac{\partial}{\partial x_b}V_d W_e = (\delta_{ad}\delta_{be} - \delta_{ae}\delta_{bd})\frac{\partial}{\partial x_b}V_d W_e$$

or $-\underbrace{\quad}$

$$= \left(\frac{\partial}{\partial x_e}V_a W_e - \frac{\partial}{\partial x_b}V_b W_a\right)$$

$\frac{\partial}{\partial x_b}$ acts on both

do derivatives

$$= \left(\frac{\partial V_a}{\partial x_e}\right)W_e + V_a\left(\frac{\partial W_e}{\partial x_e}\right) - \left(\frac{\partial V_b}{\partial x_b}\right)W_a - V_b\left(\frac{\partial W_a}{\partial x_b}\right)$$

assume $\vec{W}$ and $\vec{V}$ commute $= (\vec{W}\cdot\vec{\nabla})[\vec{V}]_a + [\vec{V}]_a(\vec{\nabla}\cdot\vec{W}) - (\vec{\nabla}\cdot\vec{V})[\vec{W}]_a - (\vec{V}\cdot\vec{\nabla})[\vec{W}]_a$

remove a's $\vec{\nabla}\times(\vec{V}\times\vec{W}) = (\vec{W}\cdot\vec{\nabla})\vec{V} + \vec{V}(\vec{\nabla}\cdot\vec{W}) - \vec{W}(\vec{\nabla}\cdot\vec{V}) - (\vec{V}\cdot\vec{\nabla})\vec{W}$ ✓

$$\vec{\nabla}\cdot(\vec{V}\times\vec{W}) = \frac{\partial}{\partial x_a}(\vec{V}\times\vec{W})_a = \frac{\partial}{\partial x_a}\varepsilon_{abc}V_b W_c = \varepsilon_{abc}\frac{\partial}{\partial x_a}V_b W_c$$

$$= \varepsilon_{abc}\left[\left(\frac{\partial V_b}{\partial x_a}\right)W_c + V_b\frac{\partial W_c}{\partial x_a}\right]$$

$\frac{\partial}{\partial x_a}$ acts on both

assume $\vec{V}$ and $\vec{W}$ commute $= \varepsilon_{cab}W_c\frac{\partial}{\partial x_a}V_b + \varepsilon_{bac}V_b\frac{\partial}{\partial x_a}W_c$

cyclic permutation $\qquad \varepsilon_{bac} = -\varepsilon_{abc}$

$$= W_c(\vec{\nabla}\times\vec{V})_c - V_b(\vec{\nabla}\times\vec{W})_b = \vec{W}\cdot(\vec{\nabla}\times\vec{V}) - \vec{V}\cdot(\vec{\nabla}\times\vec{W})$$ ✓

$$\left[\vec{\nabla}\times(\phi\vec{\nabla}\phi)\right]_a = \varepsilon_{abc}\frac{\partial}{\partial x_b}\left(\phi\frac{\partial}{\partial x_c}\phi\right) = \varepsilon_{abc}\left[\left(\frac{\partial\phi}{\partial x_b}\right)\left(\frac{\partial\phi}{\partial x_c}\right) + \phi\frac{\partial}{\partial x_b}\frac{\partial}{\partial x_c}\phi\right]$$

$= 0$ antisymmetric × symmetric in both terms.

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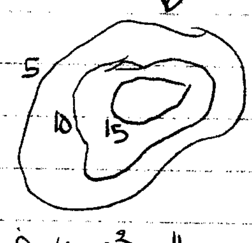
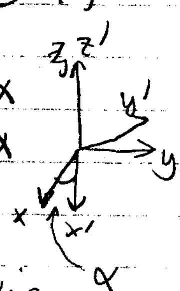
Follow-ups from Lecture 1:



- Recap on gradient $\vec{\nabla}\Phi$ (what direction?)
  - Recall that a small step $d\vec{r}$ changes Φ by $d\Phi(x,y,z) = \vec{\nabla}\Phi \cdot d\vec{r}$
    (cf. $\vec{\nabla}\Phi \cdot \Delta\vec{r} \to$ infinitesimal just means small for physicists! So $(\Delta r)^2$ small enough to neglect.)
  - Intuitive association for direction of $\vec{\nabla}\Phi$.
    Recall that force $\vec{F} = -\vec{\nabla}(\text{potential}) = -\vec{\nabla}\Phi$ along with intuition that the force is always toward lower potential ("downhill") $\Rightarrow \vec{\nabla}\Phi$ points in direction of maximum positive change ("uphill")

- How do I know that $\vec{\nabla}\Phi$ is actually a vector if $\Phi(x,y,z)$ is a scalar? (Arfken sec. 1.2)
  - Ans: Because it transforms like a vector from the x_i to x_i' coordinate systems.
  - Scalar unchanged: $\Phi(\vec{x}) = \Phi(x_i) = \Phi(x,y,z) \overset{\text{scalar}}{=} \Phi(\vec{x}') = \Phi(x_i')$
  - Define $a_{ij} = \dfrac{\partial x_i'}{\partial x_j}$, e.g. $x' = x\cos\alpha + y\sin\alpha$
    $y' = -x\sin\alpha + y\cos\alpha$
    $z' = z$



example: [i labels rows, j labels columns]

$$a_{ij} = \begin{pmatrix} \cos\alpha & \sin\alpha & 0 \\ -\sin\alpha & \cos\alpha & 0 \\ 0 & 0 & 1 \end{pmatrix} \Rightarrow \text{rotation by } \alpha \text{ about } z \text{ axis}$$

(just a special case of more general)

$\Rightarrow$ a vector transforms (ie, the coordinates change) by

$$V_i' = a_{ij}V_j \quad \text{(summation convention!)}$$

Check $\vec{V} = d\vec{x}$: $dx_i' = a_{ij}dx_j = \dfrac{\partial x_i'}{\partial x_j}dx_j$ ✓ (really start with this!)

Inverse $(\bar{a}^{-1})_{ij} = \dfrac{\partial x_i}{\partial x_j'} = a_{ji}$

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Now the dot product of two vectors should be unchanged:

Scalar $\vec{A}\cdot\vec{B} = A_\ell B_\ell \Rightarrow A_\ell \to A'_\ell = a_{\ell m} A_m,\; B_\ell \to B'_\ell = a_{\ell n} B_n$

$\Rightarrow \vec{A}'\cdot\vec{B}' = a_{\ell m} A_m a_{\ell n} B_n = a_{\ell m} a_{\ell n} A_m B_n$ but equals $\vec{A}\cdot\vec{B} = A_m B_m$

$$\Rightarrow a_{\ell m} a_{\ell n} = \delta_{nm}$$

Best to think of matrix multiplication

$a_{\ell m} = \underbrace{(a^T)_{m\ell}}_{\text{transpose}} \Rightarrow (a^T)_{m\ell} a_{\ell n} = \delta_{nm} \Rightarrow$ orthogonal matrix

$\Rightarrow \underline{\underline{a}}^T \underline{\underline{a}} = \underline{\underline{I}} \leftarrow$ identity matrix or $\underline{\underline{a}}^T = \underline{\underline{a}}^{-1}$

= means a matrix

Then $\vec{A}\cdot\vec{B}' = A_m a^T_{m\ell} a_{\ell n} B_n$
$= A_m \delta_{mn} B_n$
$= A_n B_n = \vec{A}\cdot\vec{B}$

$$\left(\ \vec{A}\ \right)\left(\ \underline{\underline{a}}^T\ \right)\left(\ \underline{\underline{a}}\ \right)\left(\vec{B}\right)$$

- $\underline{\underline{a}}$ here is real $\rightarrow$ different linear combinations of coordinates (cf. unitary transformation)

- Helmholtz Theorem (Lea 1.4)

Any vector $\vec{F}$ can be expressed as the sum of a gradient of a scalar and the curl of a vector:

$$\vec{F} = \vec{\nabla}\Phi + \vec{\nabla}\times\vec{A}$$

$\vec{\nabla}\cdot\vec{F} = \vec{\nabla}\cdot\vec{\nabla}\Phi + \cancelto{0}{\vec{\nabla}\cdot(\vec{\nabla}\times\vec{A})} = \nabla^2\Phi$ only Φ contributes to divergence

$\vec{\nabla}\times\vec{F} = \cancelto{0}{\vec{\nabla}\times\vec{\nabla}\Phi} + \vec{\nabla}\times(\vec{\nabla}\times\vec{A})$ only $\vec{A}$ contributes to curl

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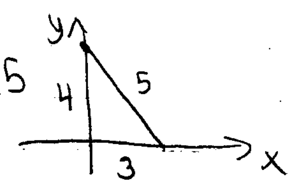
Continuations from Lecture 1 notes:

- pg. 10 Other formulas from Jackson covers: central vectors
- pg. 12 Vector calculus theorem prototype: simplest example
- pg. 14 Partial integration in vector calculus
- pg. 13 + PS#1 : Applying vector calculus theorems
  - quickly run through examples 1.2 and 1.3 from Lea Chap. 1
  - Question: What dictates the best choice of coordinates to evaluate a volume or surface integral?
    - Is it the integrand or the limits?
    - Not always a fixed answer or an optimal answer.
    - Often it is the limits → sets the geometry.
  - Comments on homework PS#1.5 and 1.6.

Prob. 1.5

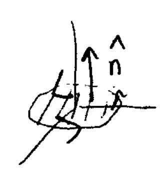


Nature of C here suggests Cartesian is by far the easiest.

- How does direction of traversing C come in?
- What is $\hat{n}$? What if you reversed the direction?
- Recall curl intuition: put small paddle wheel in moving fluid, $\vec{\nabla}\times\vec{V}$ aligned with axis tells how much it will rotate. ⇒ irrotational if $\vec{\nabla}\times\vec{V}=0$.

Prob. 1.6 Integration over a hemisphere
- Which coordinates?
- Spherical coordinate integration with Mathematica
  ⇒ go through sample notebook on Mathematica webpage

- $\hat{n}$ is positive normal to surface:
  - if closed surface, then outward
  - if open, right-hand rule on direction of boundary contour determines $\hat{n}$



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Complex Analysis

Introductory remarks:

- Complex analysis is a big subject, generally a semester course by itself in a math department
  - We'll spend about 2½ periods!
  - So we have to focus our attention and concentrate on results, not proofs

Partial list of motivations for complex analysis (see Arfken)

- real physical quantities can become complex as physical theory is made more general
  - add absortion and real index of refraction of light becomes complex
  - allow decay and real energies become complex
- $k \to ik$ relates Helmholtz to diffusion equation
- $t \to \tau = it \Rightarrow$ imaginary time equations ("Euclidean") are key to numerical solutions of field theories (and other Monte Carlo methods like Green's Function Monte Carlo)
- insight into and tools for solving differential equations
- integrals in complex plane have many applications

Core competencies

1. comfortable with manipulations of complex variables and functions
   - both Cartesian and polar form
2. evaluation of integrals. Requires:
   - understanding of analyticity (Cauchy-Riemann relations)
   - singularities (poles, branch points, essential singularities)
   - Laurent series expansion
   - specific applications of contour integration (residues et al)
3. dispersion relations (if time)

omitted: conformal mapping, proofs of any of this!

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Reminders of complex number representation ...

notation $z = x + iy \overset{\text{one to one}}{\Longleftrightarrow} (x,y)$

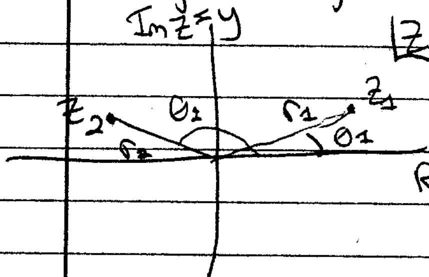
real

must know without thinking!

$\boxed{i^2=-1,\ i^3=-i,\ i^4=1}$

⇒ "Argand diagram" or "complex z plane"

$\underline{|z}$ ← notation



usually choose $0 \le \theta < 2\pi$
or $-\pi < \theta \le \pi$ so θ unique
but clearly some freedom!

$$3e^{i\pi/3} = 3e^{i(\pi/3+2\pi)} = 3e^{i(\pi/3+4\pi)} = 3e^{-5\pi/3}$$

Complex conjugate $z^* = x - iy \quad zz^* = z^*z = |z|^2 = x^2+y^2 \geq 0$

Recall how simple manipulations work through examples.

Lea, problem 2.2: Find an expression for $\cos 3\theta$ and for $\sin 3\theta$ in terms of $\cos\theta$, $\sin\theta$

Connection with complex numbers $\boxed{e^{i\theta} = \cos\theta + i\sin\theta}$ ← know results for $\theta = \frac{\pi}{2}, \pi, \frac{3\pi}{2}, 2\pi, \dots$

How do I know this? Taylor series probably best:

$$e^{i\theta} = \sum_{n=0}^{\infty} \frac{(i\theta)^n}{n!} = \sum_{n \text{ even}} + \sum_{n \text{ odd}} = \underbrace{1 - \frac{\theta^2}{2!} + \frac{\theta^4}{4!} - \dots}_{\cos\theta} \quad \checkmark$$

$$+ \underbrace{\frac{i\theta}{1!} - \frac{i\theta^3}{3!} + \frac{i\theta^5}{5!}}_{i\sin\theta}$$

Question: is $e^{iz} = \cos z + i\sin z$ for complex z?
(ans: yes, the Taylor expansion holds)

$$\Rightarrow \cos z = \frac{e^{iz}+e^{-iz}}{2}, \quad \cosh(z) = \frac{e^{z}+e^{-z}}{2} \quad \sin z = \frac{e^{iz}-e^{-iz}}{2i}, \quad \sinh z = \frac{e^{z}-e^{-z}}{2}$$

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Back to our problem ....

Take $r = 1$, $z = e^{i\theta}$

$$z^3 = (e^{i\theta})^3 = e^{i3\theta}$$

$$(\cos\theta + i\sin\theta)^3 = \cos 3\theta + i\sin 3\theta$$

expand

$$= \cos^3\theta + 3\cos^2\theta(i\sin\theta) + 3\cos(i\sin\theta)^2 + (i\sin\theta)^3 = \cos 3\theta + i\sin 3\theta$$

In an equation, real and imaginary parts must be separately equal.

[How do we know? Put all terms on one side:

(Real part) + i(Imaginary part) = 0

complex conjugate (Real part) − i (Imaginary part) = 0

⇒ add and subtract: 2 Real part = 0, $2i$ Imag part = 0

so each separately zero]

equal: real $\cos^3\theta - 3\cos\sin^2\theta = \cos 3\theta$ ✓

$3\cos^2\theta\sin\theta - \sin^3\theta = \sin 3\theta$

Is this a particularly powerful trick?

- For example, can Mathematica tell us this?
- Try these on computer

  Cos[3t] → nothing

  Cos[3t] // Simplify or // FullSimplify (do both postfix // and functional)

  ?Trig* to see choices

  Cos[3t] // TrigExpand or TrigFactor

  ⇒ works!

- Try Cos[5t] or Cos[50t] you wouldn't want to do by hand
- Try TrigToExp and ExpToTrig on

  Cos[z] + I Sin[z] and Exp[iz]
- Why does cos[3t] // TrigExpand do nothing? Need Cos, not cos (uppercase)

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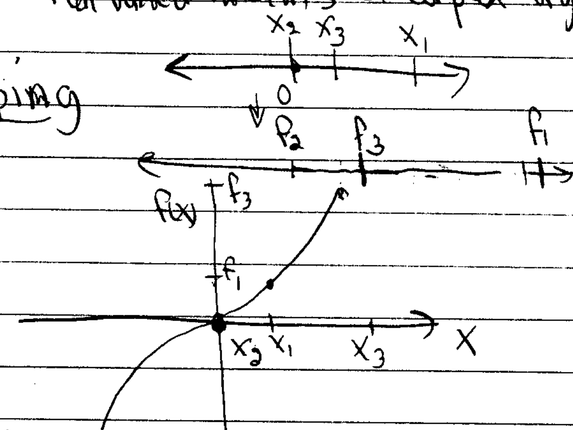
Functions of a Complex Variable

$z = x + iy \rightarrow w = f(z) = u(z) + i v(z)$

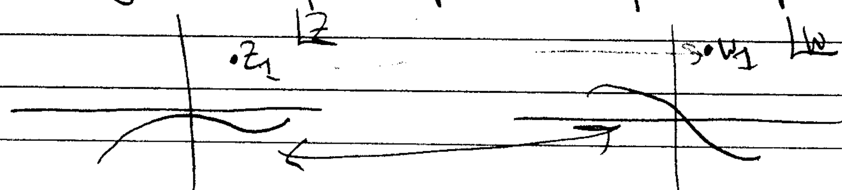
real valued functions of complex arguments

ordinary function $f(x)$ is a mapping
from $x \in \mathbb{R}$ to $f(x) \in \mathbb{R}$
"element of" real numbers

$\Rightarrow$ represent as graph



Complex $z = x + iy$ to $w = u + iv$
is mapping from complex z plane to complex w plane



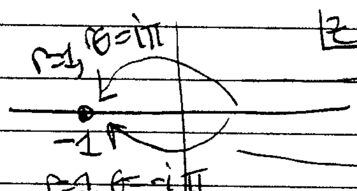
- Hard to draw an analog to $f(x)$ vs. x (need 4-dimensional representation)
- Instead, look how a line or a region maps. (more later)

Strange behavior can happen. Consider $w = z^{1/2}$

$$w = z^{1/2} = (re^{i\theta})^{1/2} = r^{1/2} e^{i\theta/2}$$

$r^{1/2}$ ← well defined positive square root

$\sqrt{-1}$?



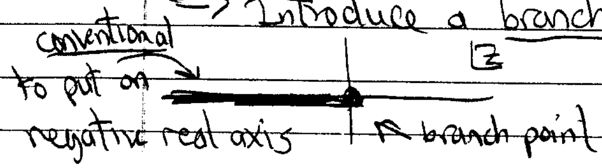
$1^{1/2} e^{i\pi/2} = +i$

$1^{1/2} e^{-i\pi/2} = -i$

} different answers depending on how z is specified!

- Looks like same point in z plane maps to two different values.

$\Rightarrow$ Introduce a branch cut to define $z^{1/2}$ as single valued.



More later! for now we've simply cut the plane so bottom and top not connected.

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Other functions with this behavior: $z^{1/n}$, $\ln z$

$$w = \ln z = \ln(re^{i\theta}) = (\ln r) + i\theta \leftarrow \text{so } \begin{array}{l} 0 \le r < \infty \\ 0 \le \theta < 2\pi \end{array}$$

only maps to a strip!

$\Rightarrow$ branch point at $z=0$, multiple sheets

Consider $z^{1/n} = (re^{i\theta})^{1/n} = r^{1/n} e^{i\theta/n} \leftarrow$ so $0 \le \theta < 2\pi n$ is mapped to $0 \le \frac{\theta}{n} < 2\pi$

$\Rightarrow$ multiple copies

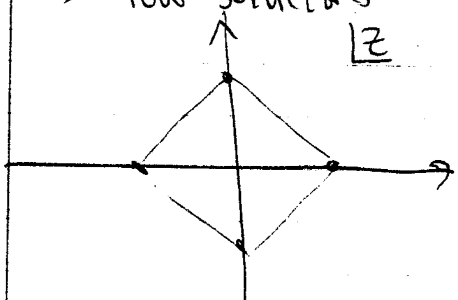
Suppose $n=4 \Rightarrow w = z^{1/4}$ or $z = w^4 \overset{\text{for example}}{=} 2$ What are solutions?

$\boxed{2 = 2e^{2\pi i n},\ n \text{ integer}}$

$\Rightarrow z$:	$2e^{i(\theta=0)}$	$2e^{2\pi i}$	$2e^{4\pi i}$	$2e^{6\pi i}$	$2e^{8\pi i}$	$2e^{10\pi i}$
w:	$2^{1/4}$	$2^{1/4}e^{i\frac{\pi}{2}}$	$2^{1/4}e^{i\pi}$	$2^{1/4}e^{i\frac{3\pi}{2}}$	$2^{1/4}e^{i2\pi}$	$2^{1/4}e^{i\frac{5}{2}\pi}$
		$\downarrow$ i	$\downarrow$ -1	$\downarrow$ $-i$	$\downarrow$ 1	$\downarrow$ i

repeats $\longrightarrow$

$\Rightarrow$ four solutions



solutions are vertices of square

$\Rightarrow$ 4 solutions to $w^4 = 2$.

for $z^{1/n}$, n-polygon starting on x-axis

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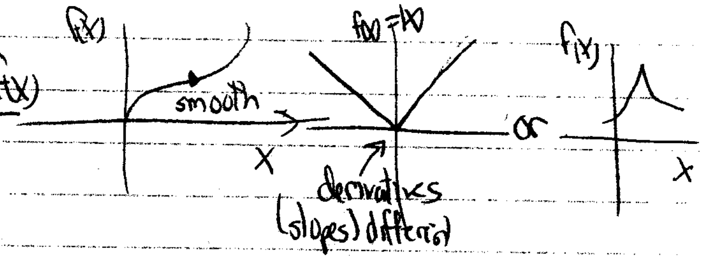
Let's put aside the tricky functions and think of nice ones: continuous and smooth.

Smooth for ordinary functions means derivatives exist from both directions:

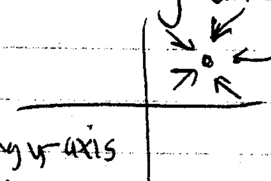
$$\left.\frac{df}{dx}\right|_{\text{at } x} = \lim_{dx\to 0} \frac{f(x+dx)-f(x)}{dx}$$



For complex functions, derivative defined similarly

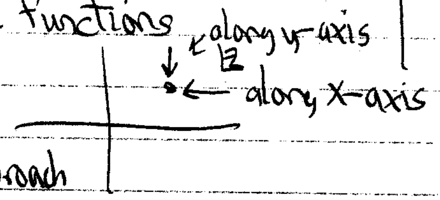
$$\frac{df}{dz} = \lim_{dz\to 0} \frac{f(z+dz)-f(z)}{dz}$$

but $z+dz$ can approach z from many directions



- requiring the same answer is a powerful constraint $\Rightarrow$ analytic functions

- Sufficient to consider just y-axis and x-axis approach



Do it

$$\frac{df}{dz} \underset{dz=(dx,0)}{\overset{x\text{-axis}}{=}} \lim_{dx\to 0} \frac{u(x+dx,y)+iv(x+dx,y)-[u(x,y)+iv(x,y)]}{dx} = \frac{\partial u}{\partial x} + \frac{i\,\partial v}{\partial x}$$

$$\underset{dz=(0,dy)}{\overset{y\text{-axis}}{=}} \lim_{dy\to 0} \frac{u(x,y+dy)+iv(x,y+dy)-[u(x,y)+iv(x,y)]}{i\,dy} = \frac{1}{i}\left(\frac{\partial u}{\partial y} + i\frac{\partial v}{\partial y}\right)$$

equate real: $\boxed{\dfrac{\partial u}{\partial x} = \dfrac{\partial v}{\partial y}}$ equate imag: $\boxed{\dfrac{\partial v}{\partial x} = \overset{\text{note!}}{-}\dfrac{\partial u}{\partial y}}$

Cauchy-Riemann equations $\Rightarrow$ powerful constraint!

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If differentiable at z_0 and nearby, then function is analytic at $z=z_0$.
- If true for all z, then function is entire

Check $f(z) = z^3$ for analyticity. Are C-R equations satisfied?

$$f(z) = z^3 = (x+iy)^3 = x^3 + 3x^2 iy + 3x(iy)^2 + (iy)^3$$
$$= \underbrace{x^3 - 3x^2y}_{u} + i(\underbrace{3x^2y - y^3}_{v})$$

$\boxed{\text{Find } u \text{ and } v \text{ first}}$

$$\frac{\partial u}{\partial x} = 3x^2 - 3y^2 \qquad \frac{\partial v}{\partial y} = 3x^2 - 3y^2 \qquad \frac{\partial u}{\partial x} = \frac{\partial v}{\partial y} \checkmark$$

$$\frac{\partial u}{\partial y} = -6xy \qquad \frac{\partial v}{\partial y} = 6xy \qquad \frac{\partial u}{\partial y} = -\frac{\partial v}{\partial y} \checkmark$$

$\Rightarrow$ C-R satisfied and partial derivatives continuous everywhere $\Rightarrow$ analytic

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Another type of problem
Find analytic function $w(z) = u(x,y) + iv(x,y)$ given that $u(x,y) = x^3 - 3xy^2$ [so find $v(x,y)$]

$$\frac{\partial u}{\partial x} = 3x^2 - 3y^2 = \frac{\partial v}{\partial y} \xRightarrow{\text{integrate}} v = 3x^2y - y^3 + f(x)$$ ← any function of x is integration constant

$$-\frac{\partial u}{\partial y} = 6xy \text{ and } \frac{\partial v}{\partial x} = 6xy + \frac{\partial f}{\partial x} \Rightarrow \text{equal if } \frac{\partial f}{\partial x} = 0 \text{ or } f(x) = \text{const}$$

$$\Rightarrow w(z) = x^3 - 3xy^2 + i(3x^2y - y^3 + \text{const.}) \text{ is analytic!}$$

Is $f(x) = \text{Re}\, z = x$ analytic?
$u = x,\ v = 0$
$\Rightarrow \frac{\partial u}{\partial x} = 1,\ \frac{\partial v}{\partial y} = 0$ not equal! So no.

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Try $f(z)=\frac{1}{z}$ for satisfying C-R equations

$$\frac{1}{z}=\frac{1}{x+iy}=\frac{1}{x+iy}\,\frac{x-iy}{x-iy}=\frac{x-iy}{x^2+y^2}\Rightarrow u=\frac{x}{x^2+y^2},\ v=\frac{-y}{x^2+y^2}$$

$$\frac{\partial u}{\partial x}=\frac{1}{x^2+y^2}+x\frac{-1}{(x^2+y^2)^2}2x=\frac{1}{(x^2+y^2)^2}\left(x^2+y^2-2x^2\right)=\frac{y^2-x^2}{(x^2+y^2)^2}$$

$$\frac{\partial v}{\partial y}=-\frac{1}{x^2+y^2}+(-y)\frac{-1}{(x^2+y^2)^2}2y=\frac{1}{(x^2+y^2)^2}\left(2y^2-(x^2+y^2)\right)=\frac{y^2-x^2}{(x^2+y^2)^2}$$

$$\Rightarrow \boxed{\frac{\partial u}{\partial x}=\frac{\partial v}{\partial y}}\ \checkmark$$

Similarly, $\frac{\partial u}{\partial y}=\frac{-x}{(x^2+y^2)^2}2y \quad \frac{\partial v}{\partial x}=\frac{+y}{(x^2+y^2)^2}2x \Rightarrow \frac{\partial u}{\partial y}=-\frac{\partial v}{\partial x}$ ✓

$\Rightarrow$ analytic <u>except</u> when $x=y=0 \Rightarrow$ at $z=0$

$f(z=0)=\infty$, called a "simple pole" (more later)

But now try $f(z)=\frac{1}{z^*}=\frac{1}{x-iy}=\frac{x+iy}{x^2+y^2}$ $\quad u=\frac{x}{x^2+y^2},\ v=\frac{y}{x^2+y^2}$

$\frac{\partial u}{\partial x}=\frac{y^2-x^2}{(x^2+y^2)^2}$, $\quad \frac{\partial v}{\partial y}=\frac{x^2-y^2}{(x^2+y^2)^2}\neq\frac{\partial u}{\partial x}$ so <u>not</u> analytic. Close doesn't count!

Can't just assign u and v functions and get analytic w!

Next: Integrals of $f(z)$!