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834 Lecture 4

Before class:

- Log into computer and set up 834 page, Mathematica page. Start up Mathematica with complex analysis notebooks.
- Ask for questions on PS#2.
- Return PS#1 — comment on results

Recap from last time plus examples:

- Solutions to complex equations. We've considered two types:

i) $f(z) = z^m = a \Rightarrow z = a^{1/m}$ with a complex and m an integer.

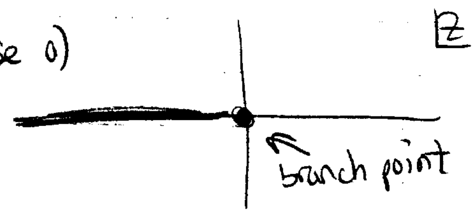
In general, write a as $a = r e^{i(\theta + 2\pi n)}$ for $n = 0, 1, 2, \ldots$ (or a variation such as n negative).

With this choice, $z = a^{1/m} = \underbrace{r^{1/m}}_{\text{unambiguous since } r \geq 0} e^{i\theta/m} \, e^{2\pi i n/m}$

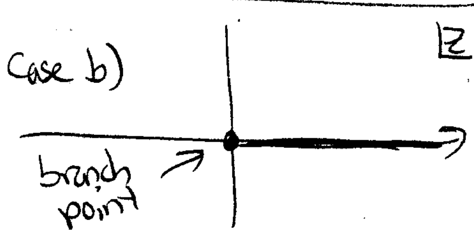
$e^{2\pi i n/m}$: m distinct possibilities, such as $n = 0, 1, 2, \ldots, m-1$. After that $\to$ duplicate values.

n labels branches; here are two choices

Case a)



Case b)



branch	range (case a)	range (case b)
Principal ($n=0$)	$-\pi < \theta \leq \pi$	$0 \leq \theta < 2\pi$
$n=1$	$\pi < \theta \leq 3\pi$	$2\pi \leq \theta < 4\pi$
⋮	⋮	⋮
$n=(m-1)$		

Note: Mathematica picks one: This can matter (integral on (43))

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ii) eg. find solutions to $\sin z = 6$

Use $\sin(x+iy) = (\sin x)(\cos iy) + (\cos x)(\sin iy)$ *

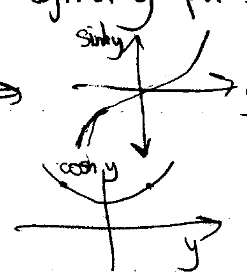
[Prove using exponentials] $= (\sin x)(\cosh y) + i(\cos x)(\sinh y) = u + iv$

equate real and imaginary parts: $u = (\sin x)(\cosh y) = 6$

$v = (\cos x)(\sinh y) = 0$ ← start here because 0.

$\sinh y = \dfrac{e^{y} - e^{-y}}{2}$ ⇒

$\cosh y = \dfrac{e^{y} + e^{-y}}{2}$



[How do I know these plots? Mathematica or consider $|y|$ small and $|y|$ Large, Then join]

← be able to do this

Either $\sinh y = 0$ or $\cos x = 0$.

If $\sinh y = 0$, then $y = 0$, $\cosh y = 1$, and $\sin x = 6$ has no solutions. ✗

So $\cos x = 0$, therefore $x = \pi/2, \frac{3\pi}{2}, \frac{5\pi}{2}$. But $\sin x$ is then $+1$ or -1, and only $+1$ has a solution.

$\Rightarrow x = \left(\frac{\pi}{2} + 2\pi n\right)$, $n = 0, \pm 1, \pm 2, \ldots$

and $y = \cosh^{-1} 6 = \pm 2.478\ldots$

↑ two solutions

* Try ArcCosh[6] in Mathematica. No numbers. Either 6. or ArcCosh[6] // N to get the decimal result.

Analyticity

If $f(z)$ is analytic in a region

- the derivative $\frac{df}{dz}$ at z_0 in region is the same from any direction $\Rightarrow \dfrac{df}{dz} = \dfrac{\partial u}{\partial x} + \dfrac{i\partial v}{\partial x} = \dfrac{1}{i}\left(\dfrac{\partial u}{\partial y} + \dfrac{i\partial v}{\partial y}\right)$

  and C-R relations.
- $f(z)$ has a convergent Taylor series about z_0.

From last time, $f(z) = z^3 \Rightarrow \frac{df}{dz} = 3z^2$ directly ← much easier!

Check: $\dfrac{\partial u}{\partial x} = 3x^2 - 3y^2$ $\quad \dfrac{\partial v}{\partial y} = 6xy$

$\Rightarrow \dfrac{\partial u}{\partial x} + i\dfrac{\partial v}{\partial x} = 3x^2 + i6xy - 3y^2 = 3(x+iy)^2 = 3z^2$ ✓

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Now back to complex integrals

$$\int_{z_0}^{z_0'} f(z)\,dz = \int (u+iv)(dx+idy) = \int_{(x_0,y_0)}^{(x_0',y_0')} [u(x,y)dx - v(x,y)dy] + i\int_{(x_0,y_0)}^{(x_0',y_0')} [v(x,y)\,dx + u(x,y)\,dy]$$

⇒ Cauchy's Theorem. $\oint_C f(z)\,dz = 0$ ← closed contour, $f(z)$ analytic on and inside

* · Complete pg. 31, Stokes's Theorem proof

· Continue with Cauchy's Integral formula $\frac{1}{2\pi i}\oint_C \frac{g(z)}{z-z_0}dz = g(z_0)$ with $g(z)$ analytic inside

* · Do pg. 32 down to QED

· What if we have $\frac{1}{2\pi i}\oint_C \frac{g(z)}{(z-z_0)^2}dz$?

$g(z)$ analytic means we can Taylor expand it:

$$\frac{1}{2\pi i}\oint \frac{g(z)}{(z-z_0)^2}dz = \frac{1}{2\pi i}\sum_{m=0}^{\infty} g^{(m)}(z_0)\oint \frac{(z-z_0)^m}{(z-z_0)^2}dz \qquad g^{(m)}(z_0) \equiv \left.\left(\frac{d^m g}{dz^m}\right)\right|_{z=z_0}$$

But recall $\oint_C (z-z_0)^n dz = \begin{cases} 0 & n \neq 1 \\ 2\pi i & n = -1 \end{cases}$

So only the $\frac{1}{2\pi i} g^{(1)}(z_0)\oint \frac{(z-z_0)^1}{(z-z_0)^2}dz = g^{(1)}(z_0)$ term survives!

General: $\boxed{g^{(m)}(z_0) = \frac{m!}{2\pi i}\oint \frac{g(z)}{(z-z_0)^{m+1}}dz}$ (but we seldom use this)

· Summary: If we have an integral $\oint_C f(z)\,dz$, it is the $\frac{1}{z-z_0}$ points that contribute.

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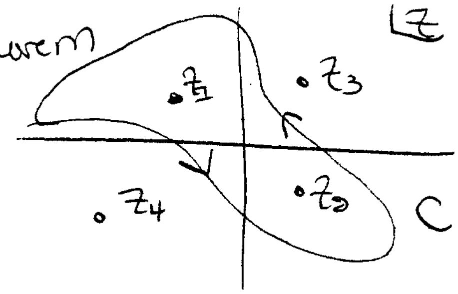
This is all we need for the residue theorem



$$\oint_C f(z)\,dz = 2\pi i \left(a_{-1,z_1} + a_{-1,z_2}\right)$$

$= 2\pi i$ (sum of "residues" of <u>enclosed</u> poles)

Here $f(z) = \sum_{n=-\infty}^{\infty} a_{n,z_i}(z-z_i)^n$ about $z_i \Rightarrow a_{-1,z_i}$ is $\frac{1}{z-z_i}$ coefficient.

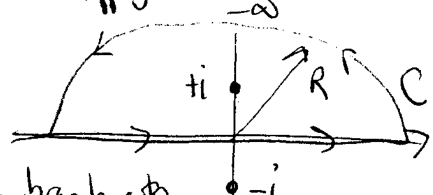
Apply this to calculate many definite integrals that arise in mathematical physics.

Aside: List of methods for definite integrals from Arfken:

1. contour integration
2. convert to gamma or beta function
3. numerical quadrature
4. integral transforms
5. series expansion and term-by-term integration

· Mathematica might use any of these methods (at your request or automagically)

Lee's steps for evaluating an integral $\rightarrow$ apply to $\int_{-\infty}^{\infty} \frac{1}{x^2+1}\,dx$ as simple example



$$\oint_C \frac{1}{z^2+1}\,dz$$

1. Draw complex z plane with contour C chosen to include integral of interest. Mark poles or other singularities, including branch cuts.
2. If there is a branch cut, "deform" the contour so it doesn't hit the cut.
3. note poles inside C (here $z=+i$)
4. evaluate the residue of f at each enclosed pole: $\frac{1}{z^2+1} = \frac{1}{z+i}\left(\frac{1}{z-i}\right) = \frac{1}{(z-i)+2i}\,\frac{1}{z-i}$
5. apply the residue theorem $\oint f\,dz = 2\pi i \,\frac{1}{2i} = \pi$
6. evaluate other non-vanishing parts (eg, integrals on both side of branch cut)

$= \frac{1}{2i}\left(\frac{1}{1+\frac{z-i}{2i}}\right)\frac{1}{z-i}$ since $\frac{z-i}{2i}$ small

$= \frac{1}{2i}\,\frac{1}{z-i}\left(1 - \frac{z-i}{2i} + \left(\frac{z-i}{2i}\right)^2 + \dots\right)$ so res $= \frac{1}{2i}$

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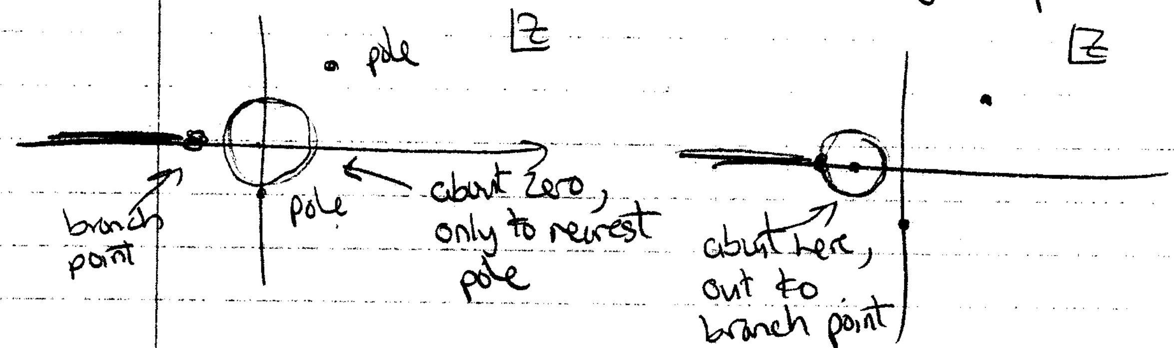
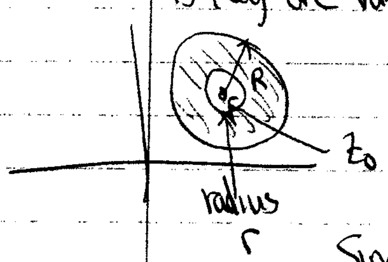
Aside on convergence of Taylor and Laurent series...

* A Taylor series about $z = z_0$ has a convergent expansion in a circle out to the first non-analytic point



- Function is defined by Taylor expansion inside "radius of convergence" (radius of circles in figures)
- converges inside, diverges outside
  - ⇒ Illustrate by Mathematica complex_series.nb notebook for one or two examples.
- extend definition further by analytic continuation (more in texts and later)

- Try Laurent series in Mathematica as well
- Derivations of Laurent series are in the book. Key generalization is they are valid in an annular region, rather than filled-in circle.



$$f(z) = \sum_{n=-\infty}^{\infty} a_n (z-z_0)^n \quad \text{unique}, \quad a_n = \frac{1}{2\pi i}\oint_c \frac{f(z)}{(z-a)^{n+1}}\, dz$$

Contour C in $r < |z-z_0| < R$ ← converge here. Often r is just z_0.

Singularities: Vocabulary

- isolated singular point if not analytic only at $z = z_0$
  - $f(z) = a_{-1}/z-z_0 + a_0 + a_1(z-z_0) + \ldots$ ⇒ simple pole with residue a_{-1}
  - if only isolated poles, then "meromorphic"
  - order n pole if $(z-z_0)^n f(z)$ is non singular at z_0 but non-zero.
- essential singularity if all n to $-\infty$ contribute: eg. $e^{1/z} = \sum_{m=-\infty}^{0} z^m \frac{1}{(-m)!}$
- branch point: multivalued $f(z)$ when circling branch point z_0

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Ways to find a residue

0. Use Mathematica Residue[f[z], {z, z_0}] (careful of branches!)

1. Find the Laurent series and pick out a_{-1} coefficient
   - often with simple Taylor expansion of non-pole piece

2. Simple pole at a:

$$\text{Res } f(a) = \lim_{z \to a} (z-a) f(z)$$

- most common rule

$$f(z) = \frac{1}{z^2+1}, \quad a = i \Rightarrow \text{Res } f(i) = \lim_{z \to i} (z-i) \frac{1}{z^2+1}$$

$$= \lim_{z \to i} \frac{1}{z+i} = \frac{1}{2i}$$

If $f(z) = \frac{\sin z}{z^2+1}$, $\Rightarrow \text{Res } f(i) = \frac{\sin i}{2i}$

3. Pole of order m $\quad \text{Res}(a) = \lim_{z \to a} \frac{1}{(m-1)!} \frac{d^{m-1}}{dz^{m-1}} \left[(z-a)^m f(z) \right]$

E.g., $f(z) = \frac{e^z}{(z-1)^2} \Rightarrow \text{Res}(z=1) = \lim_{z \to 1} \frac{1}{(2-1)!} \frac{d}{dz} \left[\cancel{(z-1)^2} \frac{e^z}{\cancel{(z-1)^2}} \right] = e^z \Big|_{z=1}$

cf. $f(z) = \frac{1}{(z-1)^2} \Rightarrow \text{Res}(z=1) = 0!$ $\qquad = e$

check Mathematica Residue[Exp[z]/(z-1)^2, {z, 1}] = e ✓

4. If $f(z) = \frac{g(z)}{h(z)}$ and $h(z)$ has simple zero at a, $g(z)$ analytic at a

$$\Rightarrow \text{Res } f(a) = \lim_{z \to a} \frac{g(z)}{h'(z)} \qquad f(z) = \tan z = \frac{\sin z}{\cos z}$$

(e.g., if non-factorizable denominator) $\qquad \text{Re } f\left(\frac{\pi}{2}\right) = \lim_{z \to \frac{\pi}{2}} \frac{\sin z}{-\sin z} = -1$

5. evaluate $\frac{1}{2\pi i} \oint_c f(z)\, dz$

- More examples from integrals later.

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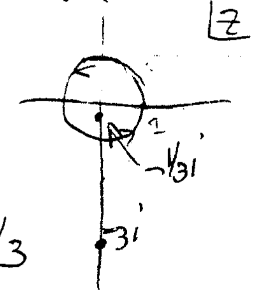
Catalog of contour integrals

① $\int_0^{2\pi} f(\sin\theta, \cos\theta)\, d\theta \Rightarrow$ strategy: take C to be unit circle in z-plane

$$z = re^{i\theta} \overset{\text{on } C}{=} e^{i\theta} \Rightarrow dz = ie^{i\theta} d\theta = iz\,d\theta \Rightarrow d\theta = \frac{dz}{iz}$$

$$\sin\theta = \frac{e^{i\theta} - e^{-i\theta}}{2i} = \frac{z - 1/z}{2i}, \quad \cos\theta = \frac{e^{i\theta} + e^{-i\theta}}{2} = (z + \tfrac{1}{z})/2.$$

$$\text{eg. } \int_0^{2\pi} \frac{d\theta}{5 + 3\sin\theta} = \oint_{\substack{\text{unit circle}\\ C}} \frac{dz/iz}{5 + 3\left(\frac{z - z^{-1}}{2i}\right)} \overset{\times\frac{2iz}{2iz}}{=} \oint \frac{2\,dz}{3z^2 + 10iz - 3}$$



quadratic formula yields simple poles $z = \frac{-10i \pm \sqrt{-100 + 36}}{6} = \frac{-10i \pm 8i}{6} = -3i, -i/3$

only $-\frac{i}{3}$ inside. $\text{Res}(-\frac{i}{3}) = \lim_{z \to -i/3} (z + \frac{i}{3}) \frac{2/3}{\cancel{(z + i/3)}(z + 3i)} = \frac{2/3}{+i(\frac{-1}{3} + 3)} = \frac{1}{4i}$

$$\Rightarrow \oint_C \frac{2\,dz}{3z^2 + 10iz - 3} = 2\pi i \left(\frac{1}{4i}\right) = \frac{\pi}{2}. \quad \text{Mathematica? } \checkmark$$

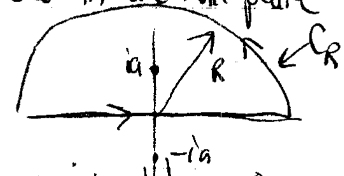
Integrate[1/(5+3Sin[t]), {t,0,2Pi}]

② $\int_{-\infty}^{\infty} f(x)\, dx$ where $f(z) \to 0$ like $\frac{1}{z^2}$ or faster

strategy: Take C to be real axis and semi-circle in one half plane

$$I = \int_{-\infty}^{\infty} \frac{1}{(x^2 + a^2)^2} dx \Rightarrow \oint_C \frac{1}{(z^2 + a^2)^2} dz = I + \lim_{R \to \infty} \int_{C_R} \frac{1}{(z^2 + a^2)^2} dz$$

$a > 0$



* always check that this doesn't contribute *

integral on C_R: $z = Re^{i\theta}$, $0 \le \theta < \pi$, as $R \to \infty$, integral $\propto \frac{R}{R^4} \times (\text{bounded integral}) \xrightarrow[R \to \infty]{} 0$

- poles of order 2 at $\pm ia \Rightarrow$ only $+ia$ in contour.

$$\text{Res } f(ia) = \lim_{z \to ia} \frac{d}{dz}\left[(z - ia)^2 \frac{1}{(z^2 + a^2)^2}\right] = \lim_{z \to ia} \frac{d}{dz} \frac{1}{(z + ia)^2} = \lim_{z \to ia} \frac{-2}{(z + ia)^3}$$

$$= \frac{-2}{(2ia)^3} = \frac{1}{4ia^3}$$

$$\Rightarrow I = 2\pi i \left(\frac{1}{4ia^3}\right) = \boxed{\frac{\pi}{2a^3}} \quad \text{Mathematica? } \checkmark$$

(with assumptions)

- What is closed in lower half plane? Then $\text{Res } f(-ia) = \lim_{z \to -ia} \frac{-2}{(z - ia)^3} = \frac{2}{(2ia)^3} = \frac{-1}{4ia^3}$

But $I = -2\pi i \cdot -1/4ia^3 = \pi/2a^3$ same answer.

clockwise contour

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③ Fourier transforms: $\int_{-\infty}^{\infty} e^{ikx} f(x)\,dx$ or $\int_{-\infty}^{\infty} \left\{ \begin{matrix} \cos kx \\ \sin kx \end{matrix} \right\} f(x)\,dx$

May need Jordan's Lemma:

If $f(z) \underset{z\to\infty}{\Longrightarrow} 0$ uniformly, then $\lim_{R\to\infty} \int_{C_R} f(z) e^{ikz}\,dz = 0$ for $k>0$ and C_R upper half of $|z|=R$ circle

$\Rightarrow$ less restrictive than $\frac{1}{z^2}$ behavior.

- For $\cos kx$ and $\sin kx$, split into $e^{\pm ikx}$ and close in appropriate half plane.

- Here an example — we'll see more later

$$I = \int_{-\infty}^{\infty} \frac{e^{ikx}}{x^4+1}\,dx \Rightarrow \oint \frac{e^{ikz}}{z^4+1} = I + \int_{C_R} \frac{e^{ikz}}{z^4+1}$$

$\to 0$ because $e^{ikz} = e^{ik(x+iy)} = e^{-ky} e^{ikx}$

$e^{-ky} \to 0$ as $y\to\infty$

because $y = R\cos t$

Poles at 4th roots of -1. $z^4 = -1 = 1\cdot e^{i\pi} e^{2\pi i n}$

$\Rightarrow e^{i\pi/4}, e^{i3\pi/4}, e^{i5\pi/4}, e^{i7\pi/4}$. First two are in upper half plane.

- details in example 2.20 in Lea

answer $I = \pi \frac{\sqrt{2}}{2} e^{-\frac{\sqrt{2}k}{2}} \left(\cos \frac{k}{\sqrt{2}} + \sin \frac{k}{\sqrt{2}}\right)$

For $\int_{-\infty}^{\infty} \cos kx\, f(x)\,dx = \mathrm{Re} \int_{-\infty}^{\infty} e^{ikx} f(x)\,dx$ usually works.

More generally, use

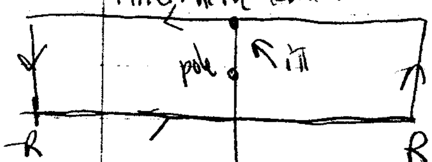
$$\int_{-\infty}^{\infty} \cos kx\, f(x)\,dx = \frac{1}{2}\int_{-\infty}^{\infty} e^{ikx} f(x)\,dx + \frac{1}{2}\int_{-\infty}^{\infty} e^{-ikx} f(x)\,dx,$$

(4) $\int_{-\infty}^{\infty} f(x)\,dx$ with hyperbolic functions (because C_R integrand doesn't vanish),

e.g. $I=\int_0^\infty \frac{\cos x}{\cosh x}\,dx \underset{\text{to }-\infty}{\overset{\text{extend}}{=}} \frac{1}{2}\int_{-\infty}^{\infty} \frac{\cos x}{\cosh x}\,dx = \frac{1}{2}\mathrm{Re}\int_{-\infty}^{\infty} \frac{e^{ix}}{\cosh x}\,dx$

Looks like $f(z)=\frac{e^{iz}}{\cosh z}$ should be good. But this doesn't vanish on our usual semicircle because of the denominator.

Alternative contour



Now we can write $\oint_C f(z)\,dz = \sum$ integrals on the sides.

The key is that the second horizontal contour is closely related to the integral on the real axis:

$$\cosh(x+i\pi) = \tfrac{1}{2}(e^x \overset{-1}{e^{i\pi}} + e^{-x}\overset{-1}{e^{-i\pi}}) = -\tfrac{1}{2}(e^x+e^{-x}) = -\cosh x$$

and

$$e^{i(x+i\pi)} = e^{ix}\cdot e^{-\pi}$$ so we get $-e^{-\pi}$ times the original integral

As shown on Lea pg. 142, the side pieces vanish. Let's check the right side:

$z = R+iy, \; 0\le y<\pi$

$$\Rightarrow \left|\int_{\text{side}} \frac{e^{iz}}{\cosh z}\,dz\right| = \left|\int_0^\pi \frac{e^{iR}e^{-y}}{e^R e^{iy}+e^{-R}e^{-iy}}\, i\,dy\right| = \frac{1}{e^R}\left|\int_0^\pi \frac{e^{iR}e^{-y}}{e^{iy}+e^{-2R}e^{-iy}}\,dy\right|$$

$$\le \frac{1}{R}\pi \to 0 \text{ as } R\to\infty$$

bounded by integral of magnitude of integrand

So we have that (in $R\to\infty$ limit)

$$\oint_{\text{rectangle}} \frac{e^{iz}}{\cosh z}\,dz = (1+e^{-\pi})\int_{-\infty}^{\infty}\frac{e^{ix}}{\cosh x}\,dx = 2\pi i(\text{sum of residues enclosed})$$

Now where is $\cosh z = 0$? $\cosh(x+iy) = \cosh x\cos y + i\sinh x\sin y \Rightarrow 0$ at $x=0$, $y = i(2n+1)\pi/2$

$\Rightarrow$ only $z = i\frac{\pi}{2}$ is inside. Residue is: $\mathrm{Res}\, f(i\tfrac{\pi}{2}) = \lim_{z\to i\frac{\pi}{2}} \frac{e^{iz}}{\sinh z} = \frac{e^{-\frac{\pi}{2}}}{\sinh i\frac{\pi}{2}} = \frac{e^{-\pi/2}}{i\sin\pi/2} = \frac{e^{-\pi/2}}{i}$

$$\Rightarrow \int_{-\infty}^{\infty}\frac{e^{ix}}{\cosh x} = \frac{1}{1+e^{-\pi}}\,2\pi i\,\frac{e^{-\pi/2}}{i} = \frac{2\pi}{e^{\pi/2}+e^{-\pi/2}} = \frac{\pi}{\cosh\frac{\pi}{2}}$$ and $$\int_0^\infty \frac{\cos x}{\cosh x}\,dx = \frac{1}{2}\mathrm{Re}\,\frac{\pi}{\cosh\frac{\pi}{2}} = \boxed{\frac{\pi}{2\cosh\frac{\pi}{2}}}$$ ✓

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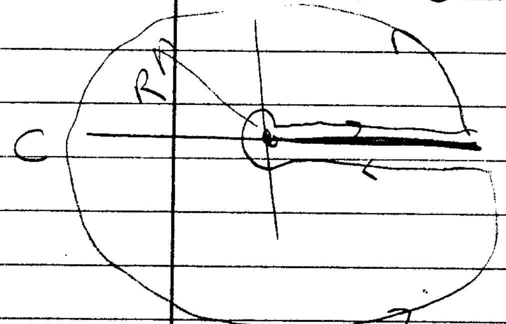
(5) $\int_0^\infty x^\alpha f(x)\,dx$ with $x \neq$ integer $\Rightarrow$ branch point at the origin.

Let's choose the branch cut along the real positive axis and the contour shown.



Why does this help? Because now the integral contour is entirely outside of the branch cut.

Let's try $I = \int_0^\infty \frac{\sqrt{x}}{x^3+1}\,dx$ (like Lea 2.23 only x^3+1 instead of x^2+1)

$f(z) = \frac{\sqrt{z}}{z^3+1}$ is analytic inside of C except at the 3 poles (cube roots of -1).

So to carry out this integral, we need to sum $2\pi i$ * the residues, check the large and small circles for contributions, and then include both the top and bottom integrals along the x-axis.

- The roots are $z^3 = e^{\pi i}\cdot e^{2\pi i n} \Rightarrow z_1 = e^{\pi/3 i},\ e^{\pi i},\ e^{\frac{5}{3}\pi i}$ (for $0 \le \theta < 2\pi$)
- The residues are $-\frac{i}{3}$, $\frac{1}{3}$, and $\frac{i}{3}$ (how do you find these?)

$$\Rightarrow \oint_C \frac{\sqrt{z}}{z^3+1} = 2\pi i\left(-\frac{i}{3} + \frac{i}{3} - \frac{i}{3}\right) = +\frac{2\pi}{3}$$

Note: Mathematica Residue command gets this wrong because different $\sqrt{x}$ branch!

- On the large circle $\left|\frac{\sqrt{z}}{z^3+1}\right| = \frac{\sqrt{R}}{R^3\left|1+e^{-3i\theta}/R^3\right|} \le \frac{1}{R^{5/2}(1-1/R^2)} \le \frac{4}{3R^{5/2}}$ for $R \ge 2$

so integral is less than $2\pi R * \frac{4}{3R^{5/2}} \xrightarrow[R\to\infty]{} 0$

- On the small circle, set $z = \varepsilon e^{i\theta}$ and take $\varepsilon \to 0$.

$$\oint_{C_\varepsilon} \frac{\sqrt{z}}{z^3+1} = i\varepsilon^{3/2}\int_{2\pi}^{0} \frac{e^{i3\theta/2}}{\varepsilon^3 e^{3i\theta}+1}\,d\theta \to 0 \text{ as } \varepsilon \to 0$$

- The upper x-axis integral is what we want.

Below we have $\theta = 2\pi$

$$\int_\infty^0 \frac{\sqrt{re^{2\pi i}}}{r^3 e^{6\pi i}+1} = -\int_0^\infty \frac{\sqrt{r}\,e^{i\pi}}{r^3+1} = \int_0^\infty \frac{\sqrt{r}}{r^3+1}\,dr = I$$

$$\Rightarrow \int_0^\infty \frac{\sqrt{x}}{x^3+1}\,dx = \frac{1}{2}\,\frac{2\pi}{3} = \frac{\pi}{3}$$