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834 Lecture 5

Before class:

- Return remaining graded PS#2's
- Start up 834 web page and mathematica but don't project
- Put contour integral plan on board
- Note that PS#3 is out and due in Dr. Prigodin's mailbox next week.
- Plan for next week: No class Wednesday? (Friday?)

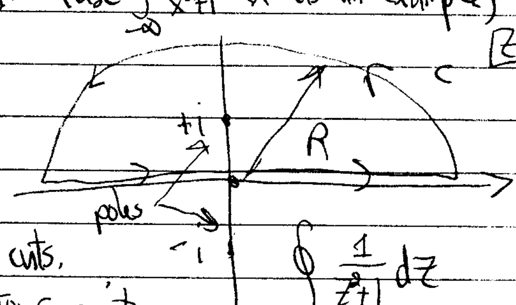
On board:

- Warm-up problem: "in your head" residue. Find residues at poles of

$$f(z) = \frac{e^{ikz}}{(z-a)(z+b)}, \qquad \begin{aligned} \operatorname{Res} f(a) &= \left[e^{ika}/(a+b) \right. \\ \operatorname{Res} f(-b) &= \left. e^{-ikb}/(-b-a) = \frac{-e^{-ikb}}{a+b} \right] \end{aligned}$$

fill in after

- Lea's steps for evaluating an integral (use $\int_{-\infty}^{\infty} \frac{1}{x^2+1}\,dx$ as an example)



$$\oint_C \frac{1}{z^2+1}\,dz$$

1. Draw complex z plane with contour C chosen to include integral of interest and rest of contour do-able. Mark poles or other singularities, including branch points and cuts.
2. If there is a branch cut, "deform" the contour so it doesn't hit the cut.
3. Notes poles inside C ("enclosed") $\Rightarrow$ here $z_0 = +i$
4. Evaluate the residue of f at each enclosed pole: $\frac{1}{z^2+1} = \frac{1}{z-i}\frac{1}{z+i} \Rightarrow \text{Res} = \frac{1}{2i}$
   a. Mathematica b. a_{-1} from Laurent series
   c. $\lim_{z\to a}(z-a)f(z)$ d. $\lim_{z\to a}\frac{1}{(m-1)!}\frac{d^{m-1}}{dz^{m-1}}\left[(z-a)^m f(z)\right]$ for order m pole
   e. $f(z) = g(z)/h(z) \Rightarrow \lim_{z\to a} g(z)/h'(z)$ e.g. $\operatorname{Re}\left(\frac{1}{\cdot}\,\frac{\sin z}{\cos z} \text{ at } \frac{\pi}{2}\right) = \frac{\sin z}{-\sin z}\Big|_{\pi/2} = -1$
   f. evaluate $\frac{1}{2\pi i}\oint_C f(z)\,dz$ where C encloses a
5. Apply the residue theorem $\oint_C f(z)\,dz = 2\pi i$ (sum residues enclosed poles)
6. Evaluate other parts or show they vanish. Here: $2\pi i \frac{1}{2i} = \pi$

$$\left|\int_{\text{semi circle}} \frac{1}{z^2+1}\,dz\right| = \left|\int_0^{\pi} \frac{1}{R^2 e^{2i\theta}+1}\, iRe^{i\theta}\,d\theta\right| \xrightarrow[R\to\infty]{} \leq \frac{1}{R}\int_0^{\pi}\left|\frac{ie^{i\theta}}{e^{2i\theta}}\right| d\theta \to 0$$

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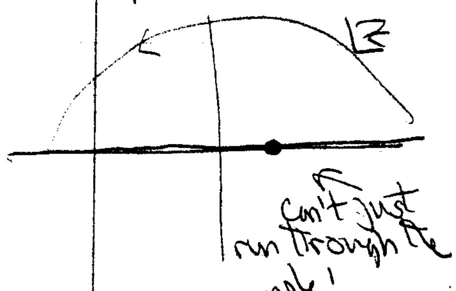
Go through other contour integral examples from pages 40, 41, 42 ⇒ start with 3, 4, 5 and revisit others if there is time.

6 Integrals with poles on the real axis.

In our examples so far along the real axis, the poles have always been somewhere in the complex plane. What if they are on the axis?

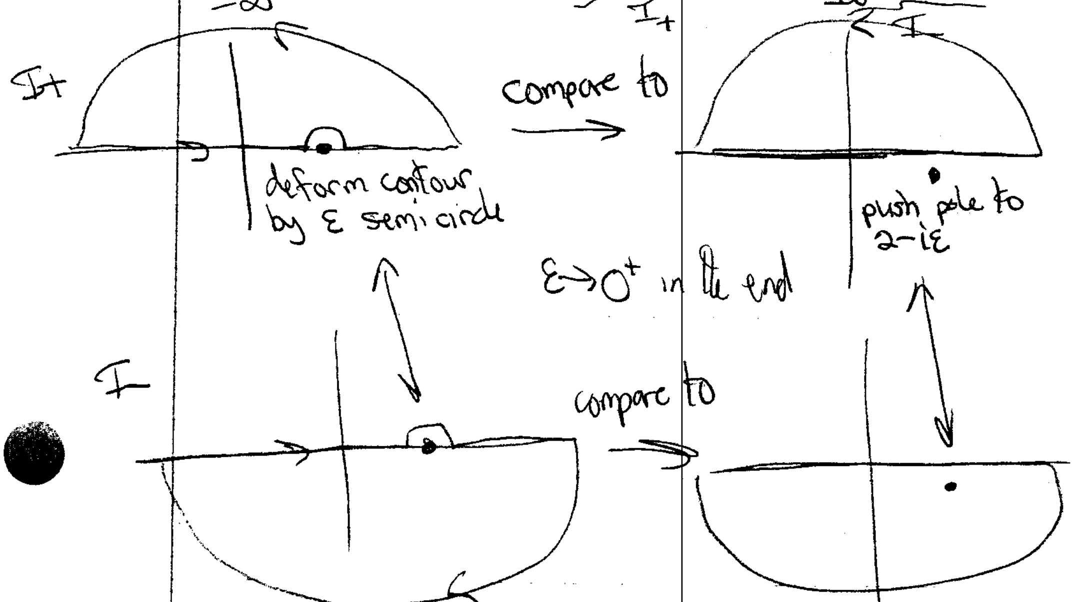


We need to specify, (usually) based on physics, how to go around the pole or how to move the pole.

Let's use the example from Lea, pg 143 $\int_{-\infty}^{\infty} \frac{\sin kx}{x-2}\,dx$ with $k>0$, but we'll move the pole.

Split into e^{ikx} and e^{-ikx} pieces to take advantage of semicircle contributions

$$\int_{-\infty}^{\infty} \frac{\sin kx}{x-2}\,dx = \frac{1}{2i}\Bigg(\underbrace{\int_{-\infty}^{\infty} \frac{e^{ikx}}{x-2}\,dx}_{I_+} - \underbrace{\int_{-\infty}^{\infty} \frac{e^{-ikx}}{x-2}\,dx}_{I_-}\Bigg) \xrightarrow[x\to z]{}$$



I_+: close in upper half plane so Jordan's lemma applies.

I_-: close in lower half plane

implied when using $i\varepsilon$ or $i\eta$ that $\lim_{\substack{\varepsilon\to 0^+ \\ \eta \to 0^+}}$ is taken. Sometimes $\frac{1}{x-2+i0^+}$ used. (46)

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⑥ (cont.)

we'll do the pushed pole

$$I_+ = \oint_{C_{R+}} \frac{e^{ikz}}{z-(2-i\varepsilon)}\,dz = 0 \quad \text{(no poles enclosed)}$$

$$= \int_{-\infty}^{\infty} \frac{e^{ikx}}{x-(2-i\varepsilon)}\,dx + \underbrace{\cancel{\oint \frac{e^{ikz}}{z-2}\,dz}}_{\text{semicircle}} \to 0$$

Jordan's lemma since $1/z-2 \xrightarrow[R\to 0]{} 0$ uniformly.

clockwise contour

$$I_- = \oint_{C_{R-}} \frac{e^{-ikx}}{z-(2-i\varepsilon)}\,dz = -2\pi i\left(e^{-ik\cdot 2}\right) \quad \text{by residue theorem}$$

$$= \int_{-\infty}^{\infty} \frac{e^{-ikx}}{x-(2-i\varepsilon)}\,dx + \cancel{\oint \frac{e^{-ikz}}{z-2}\,dz}^{\,0}$$

$$\Rightarrow \int_{-\infty}^{\infty} \frac{\sin kx}{x-(2-i\varepsilon)}\,dx \overset{\text{change variables}}{=} \int_{-\infty+i\varepsilon}^{+\infty+i\varepsilon} \frac{\sin kx}{x-2}\,dx = \frac{1}{2i}\left(0-\left(-2\pi i\, e^{-i2k}\right)\right)$$

$$= \boxed{\pi e^{-2ki}}$$

Suppose we did $\displaystyle\int_{-\infty}^{\infty} \frac{\sin kx}{x-(2+i\varepsilon)}\,dx = \int_{-\infty-i\varepsilon}^{+\infty-i\varepsilon} \frac{\sin kx}{x-2}\,dx$?

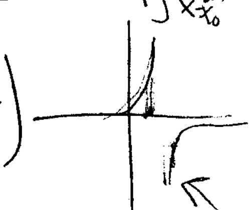
$\Rightarrow$ pick up pole in <u>upper</u> half plane $= \frac{1}{2i}\left(+2\pi i\, e^{+i2k} - 0\right)$

$$= \boxed{\pi e^{+2ki}}$$

Define <u>principal</u> <u>value</u> integral:

$$P\int_{-\infty}^{\infty} \frac{\sin kx}{x-2}\,dx \equiv \lim_{\varepsilon\to 0}\left(\int_{-\infty}^{2-\varepsilon} \frac{\sin kx}{x-2}\,dx + \int_{2+\varepsilon}^{+\infty} \frac{\sin kx}{x-2}\,dx\right)$$

$P\int \frac{1}{x-x_0}dx$



contributions cancel near pole

$$\xrightarrow[\substack{+i\varepsilon \text{ and} \\ -i\varepsilon \text{ results!}}]{\text{average of}} \frac{1}{2}\left(\pi e^{-2ki} + \pi e^{2ki}\right) = \boxed{\pi \cos 2k}$$

(like integrating above and below to remove the piece near the pole.)

- Example on PS#3.

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Brief mention: Dispersion relations

- You'll see these in E+M as the Kramers-Kronig relations, which relate the real and imaginary parts of the dielectric constant of a material ⇒ dispersive (index of refraction) and absorptive properties are related.
  - They show up in nonrelativistic and relativistic scattering.

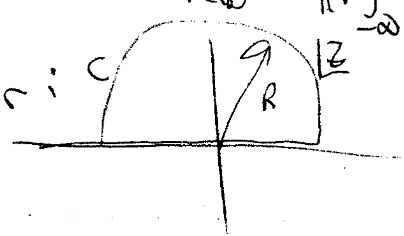
Claim: Given $f(z)$ analytic in the upper half plane with $|f(z)| \xrightarrow[z\to\infty]{} 0$ in the upper half plane ← physics tells us!

$$\mathrm{Re}[f(x_0)] = \frac{1}{\pi} P \int_{-\infty}^{\infty} \frac{\mathrm{Im}[f(x)]}{x-x_0}\,dx \qquad \text{or} \qquad u(x_0) = \frac{1}{\pi} P \int_{-\infty}^{\infty} \frac{v(x)\,dx}{x-x_0}$$

principal

$$\mathrm{Im}[f(x_0)] = -\frac{1}{\pi} P \int_{-\infty}^{\infty} \frac{\mathrm{Re}[f(x)]}{x-x_0}\,dx \qquad v(x_0) = -\frac{1}{\pi} P \int_{-\infty}^{\infty} \frac{u(x)\,dx}{x-x_0}$$

Consider Cauchy integral formula over: C



$$f(z_0) = \frac{1}{2\pi i} \oint \frac{f(z)}{z-z_0}\,dz$$

$$= \frac{1}{2\pi i} \int_{-\infty}^{\infty} \frac{f(x)}{x-z_0}\,dx + \text{vanishing semicircle}$$ if z_0 in upper half plane. If in lower half plane, the integral is zero.

Now let z_0 approach the axis from above $z_0 = x_0 + i\varepsilon$ and below $z_0 = x_0 - i\varepsilon$ and average:

$$P\int_{-\infty}^{\infty} \frac{f(x)}{x-x_0}\,dx = \frac{1}{2}\left[\int_{-\infty}^{\infty} \frac{f(x)\,dx}{x-x_0-i\varepsilon} + \int_{-\infty}^{\infty} \frac{f(x)\,dx}{x-x_0+i\varepsilon}\right] = \frac{1}{2}\left[2\pi i\, f(x_0+i\varepsilon) + 0\right] = \pi i\, f(x_0)$$

analytic so $f(x_0+i\varepsilon) = f(x_0)$ as $\varepsilon \to 0$

$$\Rightarrow \boxed{f(x_0) = \frac{1}{\pi i} P \int_{-\infty}^{\infty} \frac{f(x)}{x-x_0}\,dx}$$

Now let $f(x) = \mathrm{Re}\{f(x)\} + i\,\mathrm{Im}\{f(x)\} = u + iv$ and equate real and imaginary parts, and we're done! (The $1/i$ exchanges real ⇔ imag on right side.)

Aside: Mathematica Integrate has PrincipalValue -> True option

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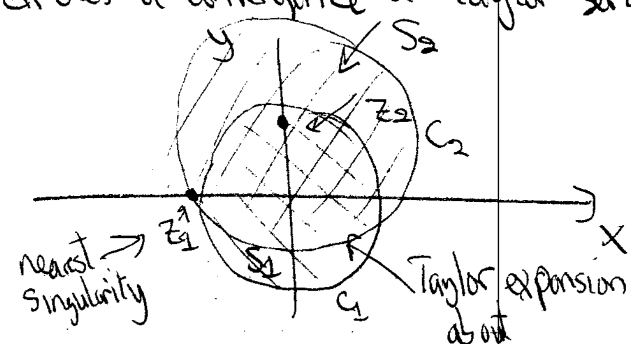
Brief mention: Analytic continuation (more later in course)

Footnote in Arfken, p434: If two analytic functions coincide (i.e. have the same value) in any region or on any line segment, then they are the same function — that is, in regions where they are both well defined, they'll give the same answer.

- This enables us to extend functions to regions in z beyond where they are originally defined.

- One way to do this analytic continuation is by overlapping circles of convergence of Taylor series.



Arfken 614

The circle of convergence C_1 is where the expansion of $f(z)$ about zero converges. So f in C_1 is defined by the Taylor series but not outside. However, there is an expansion about z_2 that is good within C_2.

- In the overlap region, f is uniquely defined, so it is the same function within both C_1 and C_2. We have "continued" the series from C_1 to C_2. And repeat.

- Alternative methods for analytic continuation will be considered later.

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Follow-up: If u and v are the real and imaginary parts of an analytic function $w(z)$, then they are harmonic: they satisfy $\nabla^2 u = 0$, $\nabla^2 v = 0$ Laplace's equations (in 2 dimensions)

Proof is simple from C-R equations:

$$\frac{\partial u}{\partial x} = \frac{\partial v}{\partial y} \quad \text{and} \quad \frac{\partial u}{\partial y} = -\frac{\partial v}{\partial x}$$

differentiate again:
$$\frac{\partial}{\partial x}\frac{\partial u}{\partial x} = \frac{\partial^2 u}{\partial x^2} \overset{CR}{=} \frac{\partial^2 v}{\partial x \partial y} \overset{\text{exchange}}{=} \frac{\partial}{\partial y}\frac{\partial v}{\partial x} \overset{CR}{=} -\frac{\partial}{\partial y}\frac{\partial u}{\partial y} = -\frac{\partial^2 u}{\partial y^2}$$

$$\Rightarrow \frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = 0 = \nabla^2 u \quad \text{(and similarly for } v\text{).}$$

An application to fluid flow is given in 2.4.1, but also relevant to E+M.

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Differential Equations: Pass 1

- In PS#3 you are asked to find solutions to Laguerre's differential equation

$$x y'' + (1-x) y' + \alpha y = 0$$

Special functions!

and the Bessel equation

$$4x^2 y'' + 4x y' + (4x^2 - 1) y = 0 \quad \text{(or divide by 4)}$$

by using series. Notation: $y = y(x)$, $y' \equiv \frac{dy}{dx}$, $y'' \equiv \frac{d^2y}{dx^2}$

- Recall the terminology (knowing these is a core competency)

  order: highest derivative $\Rightarrow$ both are 2nd order diff eqs.

  linear or nonlinear: does $y, y', y'', \ldots$ appear as more than first power? Here: no, so linear.

  $\Rightarrow$ Jackson says he will consider linear equations only.

  Nonlinear example $y y' = 2$

  Ordinary or partial: Does y depend on more than one variable (e.g. x and t) with partial derivatives for each?

  $\Rightarrow$ partial diff eq or PDE. Here, $y(x)$ so ordinary.

  - Wave equations are PDE's: $\frac{\partial^2 y(x,t)}{\partial t^2} = v^2 \frac{\partial^2 y}{\partial x^2}$

  Homogeneous or inhomogeneous:

  Does each term depend on y or derivatives $\Rightarrow$ homogeneous

$$\frac{d^2y}{dt^2} + 2\alpha \frac{dy}{dt} + \omega_0^2 y = 0 \quad \text{(damped harmonic oscillator)}$$

  - We'll come back to inhomogeneous ("driving terms" $\Rightarrow$ Green's functions, etc.)

  Are coefficients constant? (yes here)

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Many methods to solve differential equations, including numerical methods that are very important.

- Here we'll consider power series solutions. – expand around a point.

- Distinguish between expanding about a <u>regular</u> and <u>singular</u> point of the differential equation.
    - Solve for $y'' = f(x, y, y')$ [nothing but y'']

    - If homogeneous, then

$$y'' + P(x)\, y' + Q(x)\, y = 0$$

    - Cases
        i) If $P(x)$, $Q(x)$ finite at $x = x_0$, x_0 is ordinary point
        ii) If either diverge, then singular
            - <u>regular</u> if $(x-x_0)P(x)$ and $(x-x_0)^2 Q(x)$ stay finite

Bessel equation

$$x^2 y'' + x y' + (x^2 - n^2)\, y = 0$$

$$\Rightarrow \quad y'' + \frac{1}{x} y' + \left(1 - \frac{n^2}{x^2}\right) y = 0$$

$\Rightarrow P(x) = \frac{1}{x}$, $Q(x) = 1 - \frac{n^2}{x^2}$ $\Rightarrow$ $x=0$ is <u>regular</u> <u>singularity</u> and no other singular points for <u>finite</u> x.

- Almost always true in physics equations that we have no worse than a regular singular point
  $\Rightarrow$ can do series expansion

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Basic principles of series solutions

- We can expand the desired solution(s) in a series
    - it may be a Laurent series or overall fractional powers
- We can take derivatives of the series term by term (why?)
- We can equate coefficients of equal powers after we plug in the series into our equation. (why?) [uniqueness of power series]

- Solution about a regular point.
    - Do an example where we know the answer by inspection:

$$\frac{d^2y}{dx^2} + k^2 y = 0 \quad \text{(Helmholtz or simple harmoic oscillator equation)}$$

$\Rightarrow$ know $y = \sin kx$ or $\cos kx$ are solutions.

- No singular points, so $x=0$ is a regular point.

<u>Assume</u> $\quad y = \sum_{n=0}^{\infty} a_n x^n$

$$y' = \sum_{n=0}^{\infty} n a_n x^{n-1} \quad \text{(really starts at } n=1\text{)}$$

$$y'' = \sum_{n=0}^{\infty} n(n-1) a_n x^{n-2}$$

<u>Substitute</u>:

$$\sum_{n=0}^{\infty} n(n-1) a_n x^{n-2} + k^2 \sum_{n=0}^{\infty} a_n x^n = 0$$

Key: only satisfied if the net coefficient of each power of x is <u>separately</u> zero.

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Consider the first couple of x^n terms:

x^0: $n=2$ in first, $n=0$ in 2nd sum:

$$2 \cdot 1 \cdot a_2 + k^2 a_0 = 0 \Rightarrow \boxed{a_2 = -k^2 \frac{a_0}{2}}$$

x^1: $n=3$ in first, $n=1$ in 2nd sum:

$$3 \cdot 2\, a_3 + k^2 a_1 = 0 \Rightarrow a_3 = -k^2 \frac{a_1}{3 \cdot 2}$$

Generalize (we have sufficient terms to do that in this case):

x^{m-2}: $n=m$ in first, $n=m-2$ in 2nd

$$m(m-1)a_m + k^2 a_{m-2} = 0 \Rightarrow a_m = -k^2 \frac{a_{m-2}}{m(m-1)}$$

and repeat for a_{m-4}, (and so on)

$$a_m = \frac{-k^2}{m(m-1)} \frac{-k^2 a_{m-4}}{(m-2)(m-3)} = \frac{(-k^2)^2 a_{m-4}}{m(m-1)(m-2)(m-3)}$$ ← looks like factorial building up!

Continue

$$a_m = \begin{cases} (-1)^{m/2} \dfrac{k^m}{m!} a_0 & m \text{ even} \\ (-1)^{(m-1)/2} \dfrac{k^{m-1}}{m!} a_1 & m \text{ odd} \end{cases}$$

$$\Rightarrow y_1 = a_0\left(1 - \frac{(kx)^2}{2!} + \frac{(kx)^4}{4!} + \ldots\right) = a_0 \cos kx \checkmark$$

$$y_2 = a_1\left(x - \frac{k^2x^3}{3!} + \frac{k^4x^5}{5!} + \ldots\right) = \frac{a_1}{k} \sin kx$$

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Useful, but our examples seem to have singular points. ⇒ Generalizations!

If isolated, use Laurent: $\sum_{n=-m}^{\infty} a_n(x-x_0)^n$

valid for $0 < |x-x_0| < \rho$ ← radius of convergence

If not isolated (e.g. branch point), then allow for non integer values

$$y(x) = (x-x_0)^{\rho} \sum_{n=0}^{\infty} a_n (x-x_0)^n$$

ρ ← any number

⇒ Frobenius method

- power ρ of first non-vanishing term is a parameter to be determined.

See examples in Arfken 9.5 and Lea 3.3
- again, take derivatives (in special region) and substitute.
- ρ determined by finding coefficient of lowest power and setting it to zero. ⇒ indicial equation

Hypergeometric

$$(x^2-x)\frac{d^2y}{dx^2} + \left(2x-\frac{1}{2}\right)\frac{dy}{dx} + \frac{1}{4}y = 0$$

singular at $x=0,1$. Look for solution about $x_0=0$.

Substitute (next time). Lowest coefficient is $x^{\rho-1}$ with
$$-\left[\rho(\rho-1)+\tfrac{1}{2}\rho\right] a_0 = 0$$

⇒ $a_0 \neq 0$ means $\rho(\rho-\frac{1}{2})=0 \Rightarrow \rho = 0$ or $1/2$. Then match coefficients as before. (Next time)