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834 Lecture 6

Before class:

- (If available) return graded PS#2. Avg. 86 (count out of 90)
- Start up 834 web page and Mathematica but don't project
- Put principal value recap and Jordan's Lemma "Spot the Error" on the board
- Also on board: Schedule for next two weeks
  - No class Wednesday. PS#3 due at 4pm Wednesday?
  - Make-up class on Friday in Smith 1009
    - Thursday at 5pm for those with TA duties or out-of-town for APS meeting
  - Midterm on Wednesday, Oct. 19 in class on vector calculus, complex analysis, differential equations by series + asymptotic solutions
  - Short(!) PS#4 will be on remaining diff. eq. topics
  - Fourier series are next
    - "Spot the Error!" Jordan's Lemma edition

discussion: most important are core competencies: using Jackson covers, $\delta_{ij} + \epsilon_{ijk}$ manipulations, basic complex variable manipulations (e.g. solutions to equations), complex integration strategies — mostly set up, series and asymptotic solutions to diff. eqs.

Comments on homework

- New policy: identify required problems — these will be the foundation for everyone and the basis for grades (do well on these problems and exams and you get an A; these also determine minimum core competency for passing).
- Retroactive PS#2 scoring: score is out of 90 → additional is bonus
- Bonus can only help; not drag down anyone
- Use Mathematica to do (or check) algebra
- For major losses of points, see me to arrange make up.

• Come see me if $\lesssim 70$ to discuss how to fix

Lecture plan: "Spot the Error", Principal value and dispersion relations (47), (50)-(54) differential equations; if time: further example, (48) (49) topics

"Spot the Error!" from PS#2

a. $z=-1 \Rightarrow z = re^{i\theta} = -e^{i\pi/2}$

b. Laurent series about $z_0=1$: $\sum_{n=0}^{\infty} \frac{a_n z^n}{z-1}$

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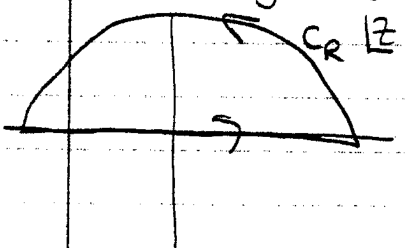
"Spot the Error!" — Jordan's lemma edition

Consider $\oint dz\, f(z) e^{ikz}$, with $k>0$, $\lim_{|z|\to\infty} f(z)=0$ on the contour

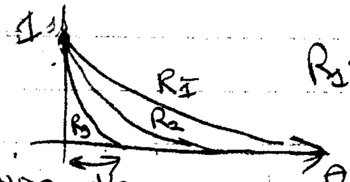


On C_R, $z=Re^{i\theta}$, $dz = iRe^{i\theta}d\theta$

$$\Rightarrow \int_{C_R} dz\, f(z)\, e^{ikz} = \int_0^{\pi} f(Re^{i\theta}) e^{ikR\cos\theta} e^{-kR\sin\theta} Rie^{i\theta} d\theta$$

Claim: As $R\to\infty$, $e^{-kR\sin\theta}$ dominates R and everything else so the integral $\to 0$.

Why is this claim incorrect?

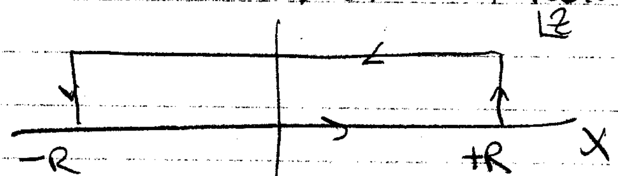


$R_1<R_2<R_3$

If we plot $e^{-kR\sin\theta}$ for different R

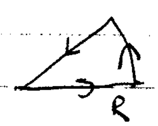
So $e^{-kR\sin\theta} \to 0$ except in a region of θ of width $1/R$ $\Rightarrow$ the contribution is $O(\frac{1}{R})$ that only just cancels the R from the measure. $\Rightarrow$ we need $f(Re^{i\theta}) \xrightarrow[R\to\infty]{} 0$ to justify Jordan's lemma. Note: Jordan's lemma with $k<0$ and closing in the lower half plane works just as well $\to$ symmetry.

- Do we have this issue with rectangular contours? No.



No, because $z=R+iy$ and $z=-R+iy$

So $\frac{e^{az}}{1+e^{bx}}$ on the right is determined by $\frac{e^{aR}}{e^{bR}}$ and on the left by e^{-aR}.

- What about $\oint e^{iz^2}dz$ on  the curved part? There's no $f(z)$ to save us!

Now $z=Re^{i\theta} \to \int_0^{\pi/4} e^{-R^2\sin 2\theta} e^{iR^2\cos 2\theta} \underline{\underline{R}}ie^{i\theta}d\theta$

and the $e^{-R^2\sin 2\theta}$ term is now $O(\frac{1}{R^2})$ in the integral so it still wins over the R.

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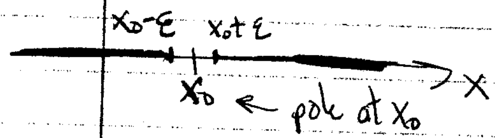
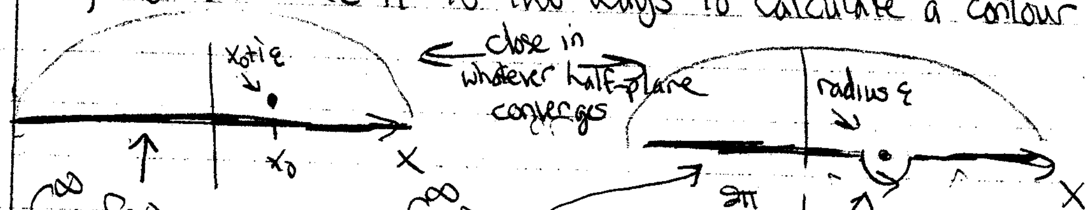
Principal value follow-up . . .

If we integrate a function with a pole on the real axis: $g(x) = \frac{f(x)}{x-x_0}$, the principal value integral is defined as
$$P\int_{-\infty}^{\infty} \frac{f(x)}{x-x_0}\,dx = \lim_{\varepsilon\to 0}\int_{-\infty}^{x_0-\varepsilon} \frac{f(x)}{x-x_0}\,dx + \int_{x_0+\varepsilon}^{\infty} \frac{f(x)}{x-x_0}\,dx$$



So we can calculate the principal value directly from this definition.

Or, we can relate it to two ways to calculate a contour



same answer for integral:
$$\int_{-\infty}^{\infty} \frac{f(x)}{x-(x_0+i\varepsilon)}\,dx = P\int_{-\infty}^{\infty}\frac{f(x)}{x-x_0}\,dx + \underbrace{\lim_{\varepsilon\to 0}\int_{\pi}^{2\pi}\frac{f(x_0+\varepsilon e^{i\theta})}{\varepsilon e^{i\theta}}\,\varepsilon i e^{i\theta}\,d\theta}_{+i\pi f(x_0)} \quad (\text{or } -\pi<\theta<0)$$

, the integrals over the semi-circles are assumed go to zero.

Symbolically: $\boxed{\frac{1}{x-x_0-i\varepsilon} = \frac{P}{x-x_0} + \pi i\,\delta(x-x_0)}$ ← $= i\pi\int_{-\infty}^{\infty} f(x)\,\delta(x-x_0)\,dx$

Similarly,

$$\int_{-\infty}^{\infty}\frac{f(x)}{x-(x_0-i\varepsilon)}\,dx = P\int_{-\infty}^{\infty}\frac{f(x)}{x-x_0}\,dx + \underbrace{\lim_{\varepsilon\to 0}\int_{\pi}^{0}\frac{f(x_0+\varepsilon e^{i\theta})}{\varepsilon e^{i\theta}}\,\varepsilon i e^{i\theta}\,d\theta}_{-i\pi f(x_0)}$$

$$\Rightarrow \boxed{\frac{1}{x-x_0+i\varepsilon} = \frac{P}{x-x_0} - \pi i\,\delta(x-x_0)}$$

Finally, $\frac{P}{x-x_0} = \frac{1}{2}\left(\frac{1}{x-(x_0+i\varepsilon)} + \frac{1}{x-(x_0-i\varepsilon)}\right)$, which we've used already

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Further comments on differential equations:

- Lea lists some methods of solution
  1. Guess the form (eg. undetermined coefficients)
  2. Power-series type solution (Frobenius) — this can be useful for numerical calculations as well
  3. Asymptotic solution
  4. ~~Relate~~ (possibly with change of variable) to known form (eg. hypergeometric equation)
  5. Integrate numerically

- Most of the rest of the course will touch on different features of differential equations!

- Sometimes Frobenius doesn't give us a 2nd solution (indicial equation has repeated root or roots differ by an integer).
  - Problem is that there is a logarithm, and Frobenius doesn't allow for it.
  - So look for

$$y_2 = \underbrace{y_1}_{\text{first solution } y_1(x)} \ln x + \sum_{n=0}^{\infty} \overbrace{b_n}^{\text{to be determined}} x^{n+p}$$

- If we want to expand around a singular point (eg. $x=1$ for the Legendre equation), simply change to $w = x-1$ so that the singularity is at $w=0$, and apply Frobenius to get $y(w)$. See Lea Example 3.7.

- The indicial equation may have complex roots. Just do it!

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Asymptotic methods $\to$ look at large values of x

- This is often helping in checking or starting numerical solutions.

Consider the example in Lea chapter 3, (example 3.9)

Modified Bessel equation is:

$$y'' + \frac{1}{x}y' - \left(1 + \frac{m^2}{x^2}\right)y = 0.$$

Find $y(x)$ as $x \to \infty$.

- For large x, the $\frac{1}{x}y'$ and $-\frac{m^2}{x^2}y$ term get small, so y_∞ satisfies

$$y''_\infty - y_\infty = 0$$

- We know the solutions: $y_\infty = (\text{const})e^{\pm x}$
- Now we can $\underline{\text{extract}}$ this dominant large x behavior by writing

$$y(x) = V(x)\,y_\infty(x) = V(x)\,e^{\pm x}$$

$$y' = V'e^{\pm x} \pm Ve^{\pm x}, \quad y'' = V''e^{\pm x} \pm 2V'e^{\pm x} + Ve^{\pm x}$$

Substituting in the equation

$$\Rightarrow V''e^{\pm x} \pm 2V'e^{\pm x} + Ve^{\pm x} + \frac{1}{x}(V'e^{\pm x} \pm Ve^{\pm x}) - \left(1 + \frac{m^2}{x^2}\right)Ve^{\pm x} = 0$$

Cancel $e^{\pm x}$ $\Rightarrow$ $V'' + \left(\frac{1}{x} \pm 2\right)V' + \frac{1}{x}\left(\pm 1 - \frac{m^2}{x^2}\right)V = 0$

Again, look at x large, assuming $V = x^\alpha(1 + \text{corrections})$.

$$\Rightarrow \alpha(\alpha-1)x^{\alpha-2} + \alpha x^{\alpha-2} \pm 2\alpha x^{\alpha-1} \pm x^{\alpha-1} - m^2 x^{\alpha-3} = 0$$

Keep leading terms: $x^{\alpha-1} \Rightarrow \pm 2\alpha \pm 1 = 0 \Rightarrow \alpha = -1/2$

$$\Rightarrow \boxed{y = \frac{e^{-x}}{\sqrt{x}}} \text{ or } \boxed{y = \frac{e^{x}}{\sqrt{x}}}$$ are the asymptotic forms.

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Frobenius Method Example

$$y(x) = x^k \sum_{n=0}^{\infty} a_n x^n$$

(A) Legendre equation: $(1-x^2)y'' - 2xy' + \alpha(\alpha+1)y = 0$

Check singularities $\Rightarrow y'' - \frac{2x}{1-x^2}y' + \frac{\alpha(\alpha+1)}{1-x^2}y = 0 \Rightarrow$ isolated singular points $x = \pm 1$

- Expand about $x=0$ (see Lea Example 3.7 for an expansion about 1)

$$y = \sum_{n=0}^{\infty} a_n x^{n+k} \qquad y' = \sum_{n=0}^{\infty}(n+k)a_n x^{n+k-1} \qquad y'' = \sum_{n=0}^{\infty}(n+k)(n+k-1)a_n x^{n+k-2}$$

Separate terms with different powers of x:

$$\sum_{n=0}^{\infty}(n+k)(n+k-1)a_n x^{n+k-2} - \sum_{n=0}^{\infty}(n+k)(n+k-1)a_n x^{n+k} - 2\sum_{n=0}^{\infty}(n+k)a_n x^{n+k} + \sum_{n=0}^{\infty}\alpha(\alpha+1)a_n x^{n+k} = 0$$

lowest powers of x: x^{k-2}, x^{k-1}, only from the first sum.

$n=0$, x^{k-2}: $k(k-1)a_0 = 0 \Rightarrow k=0$ or $k=1 \Rightarrow$ consider each.

$n=1$, x^{k-1}: $(k+1)\cdot k\, a_1 = 0$

For general term, demand that coefficient of x^{k+j}, $j \geq 0$ vanish

Set $n=j+2$ in first term and $n=j$ in others and switch to sum over j:

$$(j+2+k)(j+1+k)a_{j+2} - (j+k)(j+k-1)a_j - 2(j+k)a_j + \alpha(\alpha+1)a_j = 0$$

$\boxed{k=0}$ $(j+2)(j+1)a_{j+2} = [j(j-1)+2j - \alpha(\alpha+1)]a_j$ or $\boxed{a_{j+2} = \frac{j(j+1)-\alpha(\alpha+1)}{(j+2)(j+1)}a_j}$

$$\Rightarrow a_2 = \frac{-\alpha(\alpha+1)}{2\cdot 1}a_0, \quad a_4 = \frac{\overbrace{2\cdot 3 - \alpha(\alpha+1)}^{-(\alpha+3)(\alpha-2)}}{4\cdot 3}a_2, \quad a_6 = \frac{4\cdot 5 - \alpha(\alpha+1)}{6\cdot 5}a_4, \ldots \qquad = -(\alpha+5)(\alpha-4)$$

$$\Rightarrow \boxed{y(x) = a_0\left[1 - \frac{\alpha(\alpha+1)}{2!}x^2 + \frac{\alpha(\alpha+1)(\alpha+3)(\alpha-2)}{4!}x^4 - \frac{\alpha(\alpha+1)(\alpha+3)(\alpha-2)(\alpha+5)(\alpha-4)}{6!}x^6 + \ldots\right]}$$

even series. The general term is easy to imagine (but maybe not to write!)

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(4) (cont.)

Now we can develop an odd series starting with a_1 either from $k=0$ or $k=1$.

For $k=0$, we have the same relation between a_{j+2} and a_j, but now we start with a_1:

$$a_3 = \frac{1\cdot 2 - \alpha(\alpha+1)}{3\cdot 2} a_1, \quad a_5 = \frac{3\cdot 4 - \alpha(\alpha+1)}{5\cdot 4} a_3, \quad a_7 = \frac{5\cdot 6 - \alpha(\alpha+1)}{7\cdot 6} a_5, \ldots$$

$$1\cdot 2 - \alpha(\alpha+1) = -(\alpha+2)(\alpha-1), \qquad 3\cdot 4 - \alpha(\alpha+1) = -(\alpha+4)(\alpha-3)$$

$$\Rightarrow \boxed{y(x) = a_1\left[x - \frac{(\alpha-1)(\alpha+2)}{3!}x^3 + \frac{(\alpha-3)(\alpha-1)(\alpha+2)(\alpha+4)}{5!}x^5 + \ldots\right]}$$

For $k=1$, the recurrence is: $(j+3)(j+2)\,a_{j+2} = [(j+1)j + 2(j+1) - \alpha(\alpha+1)]a_j$

or $a_{j+2} = \dfrac{(j+1)(j+2) - \alpha(\alpha+1)}{(j+3)(j+2)} a_j$. Starting from $j=0$ gives the same series (remembering that $k=1$ means we start from x and have odd powers.

Do these series converge? Not for $x \geq 1$ for general α.
However, if α is an integer l, then the series truncates $\Rightarrow$ polynomial
Check (with $a_0=1$, $a_1=1$)

$l=0$ $\quad a_2=0$ and all higher $\Rightarrow y(x)=1$ $\quad [P_0(x)=1]$

$l=1$ $\quad a_3 \propto (1\cdot 2 - 1\cdot 2)/3\cdot 2 = 0 \Rightarrow y(x) = x$ $\quad [P_1(x)=x]$

$l=2$ $\quad a_2 = -2(3)/2$, $a_4 \propto (2\cdot 3 - 2\cdot 3) = 0 \Rightarrow y(x) = 1-3x^2$ $\quad [P_2(x) = \frac{1}{2}(3x^2-1)]$

$l=3$ $\quad a_3 = -2\cdot 5/3!$, $a_5 \propto (3\cdot 4 - 3\cdot 4) = 0 \Rightarrow y(x) = x - \frac{5}{3}x^3$ $\quad [P_3(x) = \frac{x}{2}(5x^2-3)]$

- The polynomials in [ ]'s are the Legendre polynomials, which we'll see again!
  $\Rightarrow$ They are normalized on $[-1,1]$ by $\int_{-1}^{1} P_l(x) P_{l'}(x)\,dx = \frac{2}{2l+1}\delta_{ll'}$