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834 Lecture 9

Lecture plan:

- Comments on midterm 84-85
- Fourier series recap and solution of diff. eqs. 86-87 + 89 + 81-82
- Solving S-eqn for bound states (Numerov method) 68-70
- Convergence of Fourier series 88

Before class:

- Hand back midterm
- Start up IE with 834 page. Start Mathematica with fourier_series2.nb and other pages.
  - 834_mt_histogram.pdf in solutions

On board:

- Outline of midterm comments
- Comments on PS#5 — any use of Mathematica is ok and encouraged.
  - Check the hints! Use the notebooks provided
- recap of Fourier series

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Comments on Midterm

- See histogram for distribution of scores.
  - A key has been posted with (some) partial credit guidance.
  - Please see me about any grading questions. Grading took into account time limits.
  - Primary goal: identify and fix issues identified by lower scores $\Rightarrow$ please talk to me to work out a plan to address. diagnostic

- Here: brief recap of common mistakes/misconceptions

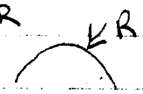
1 c) $\int_C (z-z_0)^n dz = 2\pi i$ (sum of residues) $\leftarrow$ This part ok.

- Residue $\Rightarrow$ coefficient of $\frac{1}{z-z_0} \Rightarrow$ only $n=-1$.
- all other negative n give zero, even though n^{th} order poles.
- Cf. $\int_C f(z)(z-z_0)^n dz$, eg. $\frac{f(z)}{(z-z_0)^n} = \frac{f(z_0)+(z-z_0)f'(z_0)+(z-z_0)^2\frac{f''(z_0)}{2!}}{(z-z_0)^n}$

$f(z)$ analytic $\Rightarrow$ Taylor expand!

$\Rightarrow$ eventually a term from the numerator: $\frac{(z-z_0)^{n-1}}{(z-z_0)^n}\frac{f^{(n-1)}(z_0)}{(n-1)!}$ gives a $\frac{1}{z-z_0}$ term $\Rightarrow$ residue.

1 d) $\int_{C_R} e^{iz} dz \Rightarrow$ Jordan's Lemma says $\int_{C_R} f(z)e^{iz}dz$ requires $f(z) \xrightarrow[z\to\infty]{} 0$ to vanish. Here $f(z)=1$

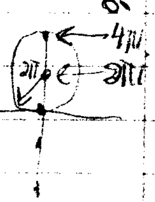


$\swarrow$ R from length

$\frac{1}{R}$ from integral (see past lectures)

$\Rightarrow O(1)$ as $R\to\infty$

6. $f(z) = \frac{z^{1/4}}{e^z-1}$ branch point + simple poles because e^z is entire



$\Rightarrow$ expand everywhere, so $(z-2\pi i n)(1+O(z-2\pi i n))$ is expansion

- essential singularity would be $e^{1/z}$ at $z=0$.
- negative integers, too!
- residue easiest to calculate with $\text{Res}(z_0) = \frac{g(z_0)}{h'(z_0)} = \frac{(2\pi i)^{1/4}}{e^{2\pi i}}$
  - how to evaluate $i^{1/4}$?
  - specify the branch, eg. $i = e^{\frac{\pi i}{2}} \Rightarrow (2\pi i)^{1/4} = (2\pi)^{1/4} e^{\pi i/8}$

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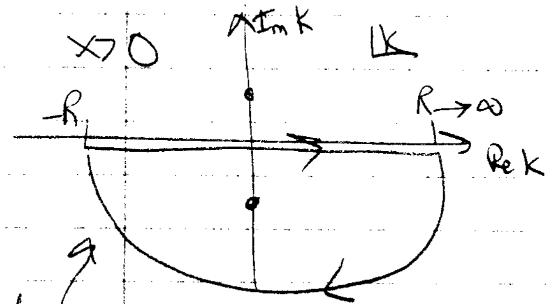
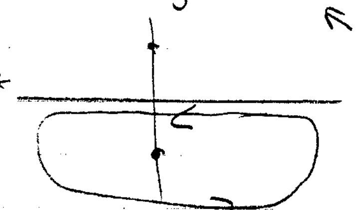
Comments on Midterm (cont.)

7. $I = \int_{-\infty}^{\infty} dk \, \frac{e^{-ikx}}{k^2+m^2}$ ⇐ an important integral!



- make sure you can clearly (and quickly) indentify which way to close

$= (-1) *$



↖ $x<0$ here is upper because e^{-ikx}. If e^{ikx}, then reversed!

Jordan's lemma is sufficient for semicircle to vanish as $R \to \infty$ ⇒ $-(2\pi i)\sum$ (residues of poles enclosed)

$x>0 \quad \frac{\pi}{m} e^{-mx} \qquad x<0 \quad \frac{\pi}{m} e^{+mx} \Rightarrow I = \frac{\pi}{m} e^{-m|x|}$

Why does it make sense that the answer is the same?
Changing $x \to -x$ is same as $i \to -i$ but $I^* = I$ because $e^{-ikx} = \cos kx + i \sin kx$ and $\cos kx$ even while $\sin kx$ odd

8. $y'' - 2xy' + 2\alpha y = 0$

$y(x) = a_0\left(1 - \alpha x^2 + \frac{\alpha(\alpha-2)}{6} x^4 + \ldots\right) + a_1\left(x - \frac{(\alpha-1)}{3} x^3 + \frac{(\alpha-1)(\alpha-3)}{30} x^5 + \ldots\right)$

so when α is an integer, only one of these series terminates: 1^{st} if α even, 2^{nd} if α odd. ⇒ Hermite polynomials

9. $y'' + \frac{1}{x} y' - y = 0 \xrightarrow{\text{large } x} y_\infty'' - y_\infty = 0 \Rightarrow y_\infty = \cancel{C_1} e^{x} + C_2 e^{-x}$

Check: $\frac{1}{x}y'$ is suppressed compared to y'' or y because y'', y', y are all same order. ⇒ self-consistent

by $v'' + v'\left(\frac{1}{x} - 2\right) - \frac{1}{x} v = 0$

For $v(x) = Cx^\alpha$, can't take $\frac{1}{x}$ terms to zero because v'', v', v are not the same order.

$v'', v'/x \propto x^{\alpha-2} \qquad v', \frac{1}{x}v \propto x^{\alpha-1}$

$\Rightarrow -2v' - \frac{1}{x}v = 0 \Rightarrow$ solve or substitute $\left((-2\alpha - 1) = 0 \Rightarrow \alpha = -\frac{1}{2}\right)$

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Fourier Series Follow-ups

Recall the generic procedure we're thinking about:
If we have a complete basis $\{\phi_n(x)\}$ then we can expand

$$f(x) = \sum_n c_n \phi_n(x)$$

with unique constant coefficients c_n.

Cf. spatial vectors $\vec{V} = \hat{x} V_1 + \hat{y} V_2 + \hat{z} V_3 \rightarrow (V_1, V_2, V_3)$

$\Rightarrow$ c_n are like the coordinates V_i

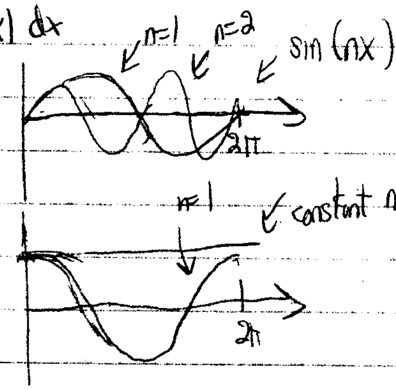
· the ϕ_n are like orthogonal vectors $\hat{x}, \hat{y}, \hat{z}$

$$\hat{x}\cdot\vec{V} = \cancelto{1}{\hat{x}\cdot\hat{x}}\, V_1 + \hat{x}\cdot\hat{y}\, V_2 + \hat{x}\cdot\hat{z}\, V_3 = V_1$$

As in QM: $$|f\rangle = \underbrace{\sum_n |\phi_n\rangle\langle\phi_n|}_{\text{completeness} \Rightarrow \text{identity}} f\rangle \Rightarrow \langle\phi_n|f\rangle = \int \phi_n^*(x) f(x)\, dx$$

$\int |x\rangle\langle x|\, dx$

· for fourier series on $x \in [0, 2\pi]$



$$f(x) = \sum_{n=0}^{\infty} a_n \sin nx + b_n \cos nx$$

$$\sin nx = \frac{e^{inx} - e^{-inx}}{2i} \qquad \cos nx = \frac{e^{inx} + e^{-inx}}{2}$$

$$= \sum_{n=0}^{\infty} \underbrace{\left(\frac{a_n}{2i} + \frac{b_n}{2}\right)}_{c_n} e^{inx} + \underbrace{\left(\frac{-a_n}{2i} + \frac{b_n}{2}\right)}_{c_{-n}} e^{-inx} = \sum_{n=-\infty}^{\infty} c_n e^{inx}$$

(note $c_n = c_{-n}^*$ if a_n, b_n are real)

and
$$a_m = \frac{1}{\pi}\int_0^{2\pi} f(x) \sin mx\, dx \quad m \geq 1$$

$$b_m = \frac{1}{\pi}\int_0^{2\pi} f(x) \cos mx\, dx \quad m \geq 1, \quad b_0 = \frac{1}{2\pi}\int_0^{2\pi} f(x)\, dx$$

$$\left.\right\} c_n = \frac{1}{2\pi}\int_0^{2\pi} f(x) e^{-inx}\, dx$$

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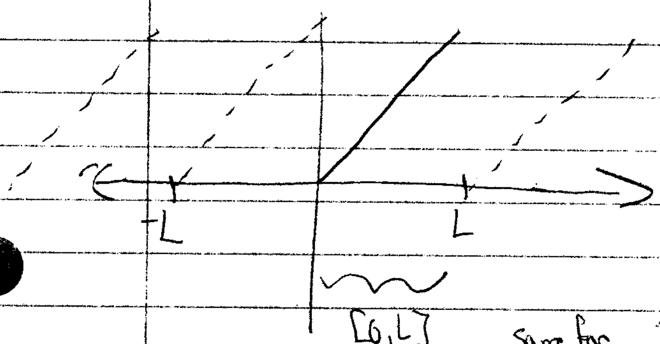
For other intervals, just change variables to get $\sin k_n x$, $\cos k_n x$, $e^{\pm i k_n x}$

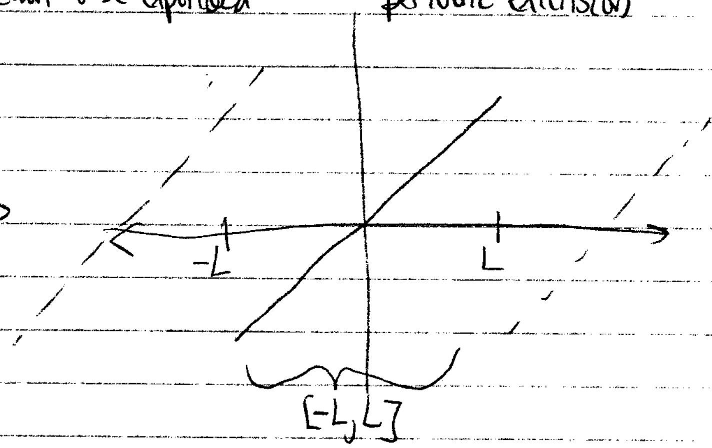
If $0<x<L$ $\quad a_m = \frac{2}{L}\int_0^L f(x)\sin\left(n\frac{2\pi x}{L}\right)dx$ $\quad$ period L

If $-L<x<L$ $\quad a_m = \frac{1}{L}\int_{-L}^{L} f(x)\sin\left(\frac{n\pi x}{L}\right)dx$ $\quad$ period $2L$

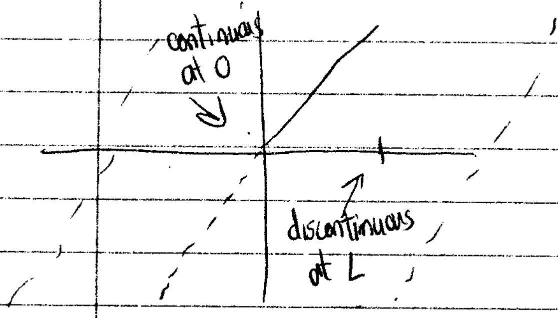
Consider $f(x)=x$ — function to be expanded - - - periodic extension



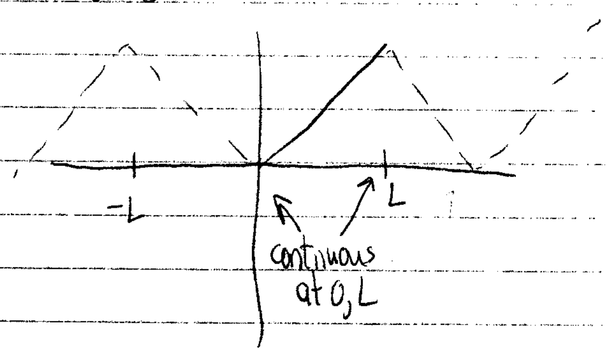
Same for odd function



Sine series



Cosine series



$f(x)=\sum_{n=1}^{\infty} a_n \sin k_n x \quad k_n=\frac{\pi n}{L}$

$a_n=\frac{2}{L}\int_0^L f(x)\sin\frac{n\pi x}{L}\,dx$

$b_n=0$

$f(x)=b_0+\sum_{n=1}^{\infty} b_n\cos\frac{n\pi x}{L}$

$b_n=\frac{2}{L}\int_0^L f(x)\sin\frac{n\pi x}{L}\,dx$

$b_0=\frac{1}{L}\int_0^L f(x)\,dx$

Looks at fourier_series2.nb examples.

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Convergence of Fourier Series

When does the Fourier series for a function $f(x)$ converge?
In general terms, if it is sufficiently smooth.

- If $f(x)$ is continuous, it will be uniformly convergent
- But if on $-L<x<L$ it is piecewise "very smooth" then it will still converge pointwise to $\frac{1}{2}[f(x^+)+f(x^-)]$ at each x (this is the average value of discontinuities that we've already seen)
- Piecewise very smooth means at least the first two derivatives exist and 2nd is continuous everywhere except for a finite number of points.

You might think intuitively that the best representation of a function by sines and cosines would depend on how many basis elements N are used.

- This would implies different sets of coefficients $\{a_n, b_n\}$ depending of N. Equivalently, that $\{c_n\}$ could be different.
- But this contradicts our contruction of the basis, which dictates unique coefficients for a given n, independent of the other n or the total N.
- So in what sense can we say we have a best fit?

Answer: In a least squares sense. If we consider the deviation

details in Lea text

$$R_N = \int_{-L}^{L} \left| f(x) - \sum_{n=-N}^{N} c_n e^{in\pi x/L} \right|^2 dx \quad \text{(sum of deviations squared)},$$

then at any N, minimizing R_N by $\frac{\partial R_N}{\partial c_k}=0$ for all k leads to $c_k = \frac{1}{2L}\int_{-L}^{L} f(x) e^{-ik\pi x/L}\,dx$, which is the usual coefficient! Also, $R_N \underset{N\to\infty}{\longrightarrow} 0$, although isolated points can differ.

Parseval's theorem: $\frac{1}{2L}\int_{-L}^{L} |f(x)|^2 dx = \sum_{n=-\infty}^{\infty} |c_n|^2 \overset{\text{physics}}{\underset{\text{RLC circuit}}{\Longrightarrow}} U = \frac{1}{\tau}\int_0^{\tau} \frac{Q(t)^2}{2C} dt = \frac{1}{2C}\sum_n |q_n|^2$ ← add up energy in each mode to get total

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Follow-up on solving inhomogeneous linear equation

$$\ddot{x} + \alpha \dot{x} + k^2 x = f(t) \quad \text{for } x(t)$$

or

charge in RLC circuit $\ddot{Q} + 2\alpha \dot{Q} + \bar{w}^2 Q = \frac{\mathcal{E}(t)}{L}$ for $Q(t)$

$\left(\frac{R}{L} = 2\alpha, \frac{1}{LC} = \bar{w}^2\right)$

- If no driving term, then constant-coefficient equation solved by e^{st} and we would find $Re(s) < 0 \Rightarrow$ damped and therefore transient.

Here: assume it has died out $\Rightarrow$ overall frequency at driving frequency of $f(t)$ or $\mathcal{E}(t)$.

- If one frequency $\mathcal{E}(t) = \mathcal{E}_0 e^{iwt}$ (can always recover $\sin(wt)$ by Im part)
- Linear so response at same w

$\Rightarrow Q(t) \propto e^{iwt}$

Plan: $\mathcal{E}(t)$ is given, so resolve into $\mathcal{E}(t) = \sum_{n=-\infty}^{\infty} \mathcal{E}_n e^{iw_n t}$, $w_n = \frac{2\pi n}{T}$ ← period

- Substitute and project each mode $Q(t) = \sum_{n=-\infty}^{\infty} q_n e^{iw_n t}$

$\Rightarrow \left(-w_m^2 + i2\alpha w_m + \bar{w}^2\right) q_m = \mathcal{E}_m / L \Rightarrow$ solve for q_m $\Rightarrow$ done!

$$q_m = \frac{\mathcal{E}_m/L}{\bar{w}^2 - w_m^2 + i2\alpha w_m} = \frac{\mathcal{E}_m/L\left(\bar{w}^2 - w_m^2 - i2\alpha w_m\right)}{\left(\bar{w}^2 - w_m^2\right)^2 + 4\alpha^2 w_m^2}$$

- $w_{-m} = -w_m$ and $(w_{-m})^2 = w_m^2$

$$\Rightarrow \left(\bar{w}^2 - w_m^2\right)\left(e^{iw_m t} + e^{-iw_m t}\right) = 2\left(\bar{w}^2 - w_m^2\right)\cos w_m t$$

$$-2i\alpha w_m \left(e^{iw_m t} - e^{-iw_m t}\right) = 4\alpha w_m \sin w_m t$$