Kronecker Delta Function δ_{ij} and Levi-Civita (Epsilon) Symbol ε_{ijk}

1. **Definitions**

$$\delta_{ij} = \begin{cases} 1 & \text{if } i = j \\ 0 & \text{otherwise} \end{cases} \qquad \varepsilon_{ijk} = \begin{cases} +1 & \text{if } \{ijk\} = 123,\ 312,\ \text{or } 231 \\ -1 & \text{if } \{ijk\} = 213,\ 321,\ \text{or } 132 \\ 0 & \text{all other cases (i.e., any two equal)} \end{cases}$$

- So, for example, $\varepsilon_{112} = \varepsilon_{313} = \varepsilon_{222} = 0$.
- The +1 (or *even*) permutations are related by rotating the numbers around; think of starting with 123 and moving (in your mind) the 3 to the front of the line, to get 312. Do it again with the 2 and you get 231. The −1 (or *odd*) permutations starting with 213 are related to each other the same way; they are related to 123 by interchanging just two of the numbers (e.g., switch the 1 and 3 to get 321).

2. **Applying δ_{ij} and ε_{ijk} to Vectors in Cartesian coordinates**

- Instead of using x, y, and z to label the components of a vector, we use 1, 2, 3.
- Then the letters i, j, k, … can be used as summation variables, running from 1 to 3. (We could use any other letters, like a, b, …; it is merely a convention.)
- Don't confuse the use of the dummy summation variables i, j, k, each of which can be 1, 2, or 3, with the unit vectors $\hat{\mathbf{i}}$, $\hat{\mathbf{j}}$, $\hat{\mathbf{k}}$. These are two independent notations!
- The dot product of two vectors $\mathbf{A}\cdot\mathbf{B}$ in this notation is

$$\mathbf{A}\cdot\mathbf{B} = A_1B_1 + A_2B_2 + A_3B_3 = \sum_{i=1}^{3} A_iB_i = \sum_{i=1}^{3}\sum_{j=1}^{3} A_iB_j\delta_{ij} \ .$$

  Note that there are nine terms in the final sums, but only three of them are non-zero.

- The i^{th} component of the cross produce of two vectors $\mathbf{A}\times\mathbf{B}$ becomes

$$(\mathbf{A}\times\mathbf{B})_i = \sum_{j=1}^{3}\sum_{k=1}^{3} \varepsilon_{ijk}A_jB_k \ .$$

  Again, there are nine terms in the sum, but this time only two of them are non-zero. Note also that this expression summarizes three equations, namely for $i = 1, 2, 3$.

3. **Einstein Summation Convention**

- We might notice that the summations in the expressions for $\mathbf{A}\cdot\mathbf{B}$ and $\mathbf{A}\times\mathbf{B}$ are redundant, because they only appear when an index like i or j appears twice on one side of an equation. So we can omit them. Thus

$$\sum_{i=1}^{3}\sum_{j=1}^{3} A_iB_j\delta_{ij} \ \longrightarrow\ A_iB_j\delta_{ij} = A_iB_i \qquad \text{and} \qquad \sum_{j=1}^{3}\sum_{k=1}^{3} \varepsilon_{ijk}A_jB_k \ \longrightarrow\ \varepsilon_{ijk}A_jB_k \ .$$

- Rules: If an index appears (exactly) twice, then it is summed over and appears only on one side of an equation. A single index (called a *free index*) appears once on each side of the equation. So

$$\textit{Valid:}\ A_i = A_j\delta_{ij}\,, \quad B_k = \varepsilon_{ikl}A_iC_l \qquad\qquad \textit{Invalid:}\ A_i = B_iC_i\,, \quad A_i = \varepsilon_{ijk}B_iC_j\ .$$

- When you have a Kronecker delta δ_{ij} and one of the indices is repeated (say i), then you simplify it by replacing the other i index on that side of the equation by j and removing the δ_{ij}. For example:

$$A_j\delta_{ij} = A_i\,, \qquad B_{ij}C_{jk}\delta_{ik} = B_{kj}C_{jk} = B_{ij}C_{ji}$$

  Note that in the second case we had two choices of how to simplify the equation; use either one!

- The triple or box product $\mathbf{A}\cdot(\mathbf{B}\times\mathbf{C})$ can be written

$$\mathbf{A}\cdot(\mathbf{B}\times\mathbf{C}) = \varepsilon_{ijk}A_iB_jC_k = \varepsilon_{kij}A_iB_jC_k = \varepsilon_{kij}C_kA_iB_j = \mathbf{C}\cdot(\mathbf{A}\times\mathbf{B})\ ,$$

  where we've used the properties of ε_{ijk} to prove a relation among triple products with the vectors in a different order.

- A very useful identity (if the repeated index is not first in both ϵ's, permute until it is):

$$\varepsilon_{ijk}\varepsilon_{ilm} = \delta_{jl}\delta_{km} - \delta_{jm}\delta_{kl}\ .$$

4. **Example: Proving a Vector Identity**

- We'll write the i^{th} Cartesian component of the gradient operator $\boldsymbol{\nabla}$ as ∂_i (cf. $\frac{\partial}{\partial x_i}$).
- Let's simplify $\boldsymbol{\nabla}\times(\boldsymbol{\nabla}\times\mathbf{A}(\mathbf{x}))$. We start by considering the i^{th} component and then we use our expression for the cross product (working from the outside in):

$$(\boldsymbol{\nabla}\times(\boldsymbol{\nabla}\times\mathbf{A}))_i = \varepsilon_{ijk}\partial_j(\boldsymbol{\nabla}\times\mathbf{A})_k\ .$$

  Next we replace the remaining cross product, making sure to introduce new dummy summation variables l and m:

$$(\boldsymbol{\nabla}\times(\boldsymbol{\nabla}\times\mathbf{A}))_i = \varepsilon_{ijk}\partial_j\varepsilon_{klm}\partial_lA_m = \varepsilon_{kij}\varepsilon_{klm}\partial_j\partial_lA_m\ .$$

  (The partial derivatives act only on the components of $\mathbf{A}$, so we can pull out the ε's.) We rotated the indices in one of the ε's in the last step so that we can now directly apply our very useful identity (and simplify):

$$(\boldsymbol{\nabla}\times(\boldsymbol{\nabla}\times\mathbf{A}))_i = (\delta_{il}\delta_{jm} - \delta_{im}\delta_{lj})\partial_j\partial_lA_m = \partial_m\partial_iA_m - \partial_l\partial_lA_i = \partial_i(\partial_mA_m) - (\partial_l\partial_lA)_i$$

  or, finally,

$$\boldsymbol{\nabla}\times(\boldsymbol{\nabla}\times\mathbf{A}) = \boldsymbol{\nabla}(\boldsymbol{\nabla}\cdot\mathbf{A}) - \nabla^2\mathbf{A}\ .$$