## Physics 834: Problem Set #10

These problems are due in Dr. Vladimir Prigodin's mailbox in the main office by 4pm on Thursday, December 1. Check the 834 webpage for suggestions and hints. Please give feedback early and often (and email or stop by M2048 to ask about anything).

There are two groups of problems. The first group is required of everyone; if you do these correctly you will get 100% of the points for the problem set. The second group is optional but is recommended to go into greater depth in the material, if you have time. These will be awarded bonus points.

**Required problems**

1. (*20 pts*) We may model the force between particles in an atomic nucleus with a three-dimensional square-well potential

$$V(r) = \begin{cases} -V_0 & \text{for } 0 \le r < R \\ 0 & \text{for } r > R \end{cases} \tag{1}$$

Schrödinger's equation for this system takes the form

$$\left(\nabla^2 - 2\frac{m}{\hbar^2}V(r)\right)\psi(\mathbf{x}) = -2\frac{m}{\hbar^2}E\psi(\mathbf{x}) \;, \tag{2}$$

where the particle's energy $E$ is negative.

   (a) Write the differential operator in spherical coordinates and show that the general solution inside and outside the square well may be written in terms of spherical Bessel functions. We require $\psi(0)$ be finite and $\psi(\mathbf{x}) \to 0$ as $x \to \infty$.

   (b) With $\alpha^2 \equiv 2(m/\hbar^2)V_0R^2$ and $\epsilon = -E/V_0$ with $E < 0$, show that the energy levels are determined by the equation

$$\sqrt{1-\epsilon}\, k_l(\alpha\sqrt{\epsilon})\, j_{l+1}(\alpha\sqrt{1-\epsilon}) = \sqrt{\epsilon} j_l(\alpha\sqrt{1-\epsilon}) k_{l+1}(\alpha\sqrt{\epsilon}) \;, \tag{3}$$

   where $k_l$ is a modified spherical Bessel function (see Arfken problems 11.7.15 through 11.7.21). The boundary condition at $r = R$ is that the wave function and its radial derivative be continuous there.

   (c) Use Mathematica to find the energy of the lowest energy level for $l = 0$, $\alpha^2 = 10$. Attach a printout.

2. (*20 pts*) Consider the damped harmonic oscillator:

$$\frac{d^2y}{dt^2} + 2\alpha\frac{dy}{dt} + \omega_0^2 y = f(t) \;. \tag{4}$$

You can assume that $\omega_0^2 > \alpha^2$ but comment on other possibilities.

   (a) Apply the division-of-region method to find the Green's function.

(b) Use it to find the response of the oscillator to the input $f(t) = 1 - t/T$ for $0 < t < T$ and zero otherwise.

**Optional problems (counts as bonus points)**

3. (*20 pts*) Consider the Neumann Green's function for the one-dimensional Helmholtz equation

$$\frac{d^2\Phi}{dx^2} + k^2\Phi(x) = -\frac{\rho(x)}{\epsilon_0} \tag{5}$$

with boundary conditions $dG/dx = 0$ at $x = 0$ and $x = L$.

(a) Find this Green's function using the division-of-region method.

(b) Find this Green's function as a series of eigenfunctions.

(c) What happens in the $k \to 0$ limit, when the Helmholtz equation reduces to the one-dimensional Poisson equation?

(d) Write an integral for the potential $\Phi(x)$ if

$$\rho(x) = \begin{cases} \rho_0 x/L & \text{if } 0 < x < L/2 \\ 0 & \text{otherwise} \end{cases} \tag{6}$$

and $d\Phi/dx = 0$ at $x = 0$ and $x = L$. Evaluate $\Phi(x)$ and plot it in Mathematica (and print out your notebook) for $L = \epsilon_0 = \rho_0 = 1$, $k = 0.5$.

4. (*10 pts*) Consider the diffusion equation

$$\frac{\partial G}{\partial t} - D\frac{\partial^2 G}{\partial x^2} = \delta(x - x')\delta(t - t') \tag{7}$$

for $-\infty < x < \infty$ and $0 \le t < \infty$.

(a) Find the Green's function for this equation by taking the Fourier transform in space and using the division-of-region method in time.

(b) Use your result to compute the function $f(x, t)$ that satisfies the diffusion equation

$$\frac{\partial f}{\partial t} - D\frac{\partial^2 f}{\partial x^2} = S(x, t) \tag{8}$$

with the source function

$$S(x, t) = e^{-x^2/a^2}\delta(t) \ . \tag{9}$$

5. (*10 pts*) Find the Dirichlet Green's function for Poisson's equation in the interior of a hemisphere of radius $a$.

(a) Choose $0 \le \phi \le \pi$ and $0 \le \theta \le \pi$.

(b) Choose $0 \le \theta \le \pi/2$ and $0 \le \phi \le 2\pi$.

(c) Use one of these two Green's functions to evaluate the potential inside the hemisphere if $\Phi = 0$ on the spherical surface and $\Phi(r) = V_0(1 - r/a)$ on the flat face.