Physics 834: Problem Set #2

These problems are due in class on Wednesday, October 5. Check the 834 webpage for suggestions and hints. Please give feedback early and often (and email or stop by M2048 to ask about anything).

1. (*15 pts*) **Functions in the complex z plane.**

   (a) Find *all* solutions to $z^5 = -1$ and plot in complex plane.

   (b) Find *all* solutions to $\cos z = 100$.

   (c) Show that $|\sin z| \geq |\sin x|$ for all z, where x is the real part of z.

2. (*15 pts*) Investigate the function $w = 1/\sqrt{z}$.

   (a) Find the real-valued functions $u(r,\theta)$ and $v(r,\theta)$, where $w = u + iv$.

   (b) How many branches does this function have?

   (c) Find the image of the unit circle under this mapping.

3. (*10 pts*) Small amplitude waves in a plasma are described by the relations

$$\frac{\partial n}{\partial t} + \frac{\partial}{\partial x}(n_0 v) = 0 \tag{1}$$

$$\epsilon_0 \frac{\partial E}{\partial x} = -en \tag{2}$$

$$m\frac{\partial v}{\partial t} = -eE - m\nu v\ , \tag{3}$$

   where n_0, e, m, ν, and ϵ_0 are constants. The constant ν is the collision frequency. Assume that n, E, and v are all proportional to $\exp(ikx - i\omega t)$.

   (a) Solve the equations for nonzero n, E, and v to show that ω satisfies the equation

$$\omega^2 + i\nu\omega = \frac{n_0 e^2}{m\epsilon_0} \equiv \omega_p^2\ , \tag{4}$$

   where ω_p is the plasma frequency.

   (b) Solve this equation to find the frequency ω and explain how this shows that collisions damp the waves.

4. (*20 pts*) **Cauchy-Riemann relations.**

   (a) Find the analytic function $w(z) = u(x,y) + iv(x,y)$ if $v(x,y) = e^{-y}\sin x$.

   (b) For each of the following complex functions, find the real and imaginary parts u and v (i.e., $f = u + iv$) and show that u and v obey the Cauchy-Riemann relations. Then find the derivative df/dz directly by differentiating with respect to z and *also* by differentiating with respect to x and y and expressing the answer in terms of z. Do they agree?

i. $f = z^2 \sin z$

ii. $f = \dfrac{1}{1+z}$

(c) One of the functions $u_1 = 2(x-y)^2$ and $u_2 = \frac{x^3}{3} - xy^2$ is the real part of an analytic function $w(z) = u + iv$. Which is it? Find the function $v(x, y)$ and write w as a function of z.

5. (*15 pts*) Determine the Taylor or Laurent series for each of the following functions in the immediate neighborhood of the point specified. This means to find the general term (as a function of an index n) and not just the first few terms. In each case, determine the region of convergence of the series. Can you check your answers with Mathematica?

(a) $\dfrac{\cos z}{z-1}$ about $z = 1$

(b) $\dfrac{\sin z^2}{z}$ about $z = 0$

(c) $\dfrac{\ln z}{z-1}$ about $z = 1$

6. (*15 pts*) Find the residue of each of the following functions at the point specified.

(a) $\dfrac{z-2}{z^2-1}$ at $z = 1$

(b) $\dfrac{\sin z}{z^2}$ at the origin

(c) $\dfrac{\cos z}{1/2 - \sin z}$ at $z = \pi/6$

7. (*10 pts*) Evaluate the following integrals.

(a) $\oint_C \frac{\cos z}{z}\, dz$ where C is a circle of radius 2 centered at the origin.

(b) $\int_{-\infty}^{+\infty} \dfrac{1}{x^2+2}\, dx$ by the residue theorem.