

Physics 834: Problem Set #5

These problems are due in Weishi (Shirley) Li's mailbox in the main office by 4pm on Wednesday, October 26. Check the 834 webpage for suggestions and hints. Please give feedback early and often (and email or stop by M2048 to ask about anything).

There are two groups of problems. The first group is required of everyone; if you do these correctly you will get 100% of the points for the problem set. The second group is optional but is recommended to go into greater depth in the material, if you have time. These will be awarded bonus points.

Required problems

- (15 pts) Develop the full Fourier series for the function $f(x) = x$
 - over the range $0 \leq x \leq 1$;
 - over the range $-1 \leq x \leq 1$.
 - Use Mathematica to make a plot (attach a printout) showing the original function and the sum of the first three nonzero terms in each series. Comment on the similarities and differences between the two series.
- (10 pts) Develop the Fourier series for the function $f(x) = x^2$ over the range $0 \leq x \leq 1$ and make/attach a Mathematica plot as in problem 1.
- (10 pts) Find an exponential Fourier series for the function $\sinh \alpha x$ on the range $0 \leq x \leq 2\pi$. By combining terms, rewrite your answer as a series in sines and cosines.
- (10 pts) A spring-and-dashpot system satisfies the equation

$$\frac{d^2x}{dt^2} + \alpha \frac{dx}{dt} + k^2x = f(t). \quad (1)$$

The system is driven by a periodic driving force with period T :

$$f(t) = \begin{cases} at & \text{if } 0 < t \leq T/2 \\ a(T-t) & \text{if } T/2 \leq t < T \end{cases} \quad (2)$$

Find the response of the system $x(t)$ as a Fourier series.

- (15 pts) A guitar string of length $L = 65$ cm is plucked by pulling it to the shape

$$y(x, 0) = \begin{cases} ax^2 & \text{if } 0 < x < L/3 \\ (a/4)(L-x)^2 & \text{if } L/3 < x < L \end{cases} \quad (3)$$

and then letting go.

- Determine the subsequent motion of the string (that is, find $y(x, t)$ for $t > 0$).
- Which harmonics are excited? Why? (Contrast to plucking in the middle.)

- (c) Use Mathematica to plot the original string shape and then the string displacement as a function of x for $t = 0, 0.4,$ and 0.8 times L/v (attach your plot). Comment on the plots.

Optional problems (counts as bonus points)

6. (5 pts) A function $f(x)$ is expanded in an exponential Fourier series

$$f(x) = \sum_{n=-\infty}^{\infty} c_n e^{inx} \quad (4)$$

If $f(x)$ is real, $f(x) = f^*(x)$, what restriction is imposed on the coefficients c_n ?

7. (5 pts) Use the Fourier series for the step function to evaluate the sum

$$\sum_{m=0}^{\infty} \frac{(-1)^m}{2m+1}. \quad (5)$$

Use Parseval's theorem applied to the same series to obtain the sum

$$\sum_{m=0}^{\infty} \frac{1}{(2m+1)^2} \quad (6)$$

8. (10 pts) A rectangular box measuring $a \times b \times c$ has all its walls at temperature T_1 except for the one at $z = c$, which is held at temperature T_2 . When the box comes to equilibrium, the temperature function $T(x, y, z)$ satisfies the equation

$$\frac{\partial T}{\partial t} = D \nabla^2 T \quad (7)$$

with the time derivative on the left equal to zero. Using the method in Lea, Example 3.15, find the temperature T in the box in the form

$$T(x, y, z) = T_1 + \tau(x, y, z) \quad (8)$$

where τ is expressed in a Fourier series

$$\tau(x, y, z) = \sum_{n,m} a_{nm} \sin \frac{n\pi x}{a} \sin \frac{m\pi y}{b} f(z) \quad (9)$$

Find the function $f(z)$ and the coefficients a_{mn} .