

Physics 834: Problem Set #8

These problems are due in Weishi (Shirley) Li's mailbox in the main office by 4pm on Thursday, November 17, 2011. Check the 834 webpage for suggestions and hints. Please give feedback early and often (and email or stop by M2048 to ask about anything).

There are two groups of problems. The first group is required of everyone; if you do these correctly you will get 100% of the points for the problem set. The second group is optional but is recommended to go into greater depth in the material, if you have time. These will be awarded bonus points.

Required problems

1. (25 pts) Consider the three-dimensional wave equation

$$\frac{\partial^2 s}{\partial t^2} - v^2 \nabla^2 s = f(\mathbf{x}, t). \quad (1)$$

- (a) Take the Fourier transform of this equation and solve for the transform $S(\mathbf{k}, \omega)$.
- (b) Show that the introduction of a damping force (through the addition of a term $\alpha \partial s / \partial t$ on the left-hand side) moves the poles off the real axis.
- (c) Invert the transform in the case $\alpha \rightarrow 0^+$ for $f(\mathbf{x}, t) = e^{-r/a} \delta(t)$, where r is distance from the origin and a is a positive constant.
- (d) Invert the transform in the case $\alpha \rightarrow 0^+$ for $f(\mathbf{x}, t) = \delta(\mathbf{x}) \delta(t)$.

2. (25 pts) The Helmholtz equation

$$\frac{d^2 y}{dx^2} + k^2 y = 0 \quad (2)$$

is subject to the boundary conditions

$$ay + by' = 0 \quad (3)$$

at $x = 0$ and

$$\alpha y + \beta y' = 0 \quad (4)$$

at $x = L$.

- (a) Find the specific forms of the eigenfunctions.
 - (b) Obtain eigenvalues for the case $a = \alpha$ and $b = \beta$
 - (c) Obtain eigenvalues for the case $aL = b$ and $\beta = 2\alpha L$
3. (25 pts) A set of eigenfunctions $y_n(x)$ satisfies the Sturm-Liouville equation

$$\frac{d}{dx} \left(f(x) \frac{dy}{dx} \right) - g(x)y + \lambda w(x)y = 0 \quad (5)$$

with boundary conditions

$$\alpha_1 y + \beta_1 \frac{dy}{dx} = 0 \quad \text{at } x = a \quad (6)$$

$$\alpha_2 y + \beta_2 \frac{dy}{dx} = 0 \quad \text{at } x = b \quad (7)$$

$$(8)$$

The function $g \equiv 0$.

- (a) Show that the derivatives $u_n(x) = y'_n(x)$ are also orthogonal functions.
 - (b) Determine the weighting function for these functions.
 - (c) What boundary conditions are required for orthogonality?
 - (d) Apply your results to the Legendre equation to determine the orthogonality of the derivatives $P'_l(\mu)$.
4. (25 pts) Use the recursion relations for the Legendre polynomials $P_l(x)$ to evaluate the integral

$$I_l \equiv \int_{-1}^{+1} \frac{P_l(x)}{\sqrt{1-x^2}} dx \quad (9)$$

and hence obtain a Fourier-Legendre series for the function $1/\sqrt{1-x^2}$.

Optional problems (counts as bonus points)

5. (20 pts) A long beam is resting on an elastic foundation. The equation satisfied by the beam displacement, called the Euler-Bernoulli equation, is

$$EI \frac{d^4 y}{dx^4} = q(x) - \alpha y(x) \quad (10)$$

where $q(x)$ is the load and α is a constant describing the elastic properties of the foundation. The product EI is also constant (E is the elastic modulus and I is the second moment of area). If the load is concentrated toward the center of the beam, then we may assume that $y \rightarrow 0$ as $x \rightarrow \pm\infty$.

- (a) Transform the equation and find $Y(k)$ in terms of $Q(k)$.
 - (b) Solve for the beam displacement if $q(x) = Mg\delta(x-a)$.
 - (c) Solve for the beam displacement if $q(x) = (Mg/L)[\theta(x-a+L/2) - \theta(x-a-L/2)]$.
6. (10 pts) Sum the series

$$\sum_{p=0}^{\infty} (-1)^p \frac{2p+1}{x^2 + (2p+1)^2} \quad (11)$$

by taking the Fourier transform of each term, summing the series in the transform space, and then transforming back.