Physics 834: Problem Set #9

These problems are due in Weishi (Shirley) Li's mailbox in the main office by 4pm on Wednesday, November 23. Check the 834 webpage for suggestions and hints. Please give feedback early and often (and email or stop by M2048 to ask about anything).

There are two groups of problems. The first group is required of everyone; if you do these correctly you will get 100% of the points for the problem set. The second group is optional but is recommended to go into greater depth in the material, if you have time. These will be awarded bonus points.

Required problems

1. (*20 pts*) Find the electrostatic potential inside a hemisphere of radius a with potential $\Phi = 0$ on the flat side and $\Phi = V_0$ on the curved part.

2. (*20 pts*) Use the generating function

$$g(t,x) = \frac{1}{\sqrt{1-2xt+t^2}} = \sum_{0}^{\infty} P_n(x) t^n \tag{1}$$

and the addition theorem for spherical harmonics

$$P_l(\cos\gamma) = \sum_{m=-l}^{l} \frac{4\pi}{2l+1} Y_{lm}(\theta,\phi) Y^*_{lm}(\theta',\phi') \tag{2}$$

   (a) to derive the expansion

$$\frac{1}{|\mathbf{x}-\mathbf{x}'|} = \sum_{l=0}^{\infty} \sum_{m=-l}^{l} \frac{4\pi}{2l+1} \frac{r_<^l}{r_>^{l+1}} Y_{lm}(\theta,\phi) Y^*_{lm}(\theta',\phi')\ , \tag{3}$$

   where $r_<$ and $r_>$ are the lesser and the larger of r and r' respectively.

   (b) Use this expansion to find the magnetic vector potential due to a circular loop of wire of radius a that is carrying current I.

3. (*20 pts*) A solid sphere of radius a is immersed in a vat of fluid at temperature T_0. Heat is conducted into the sphere according to

$$\frac{\partial T}{\partial t} = D\nabla^2 T\ . \tag{4}$$

   If the temperature at the boundary is fixed at T_0 and the initial temperature of the sphere is T_1, find the temperature within the sphere as a function of time.

Optional problems (counts as bonus points)

4. (*20 pts*) Current flow in a conducting sheet is described by the relations $\mathbf{D} = -\boldsymbol{\nabla}\Phi$ and $\mathbf{j} = \sigma\mathbf{D}$.

(a) Use the charge conservation law ($\partial \rho / \partial t + \boldsymbol{\nabla} \cdot \mathbf{j} = 0$) to show that, in a steady state, Φ satisfies Laplace's equation.

(b) Find the eigenfunctions for current flow in a circular copper plate.

(c) Current I flows into and out of a plate of thickness t and conductivity σ through electrodes at $r = a$ extending from $\theta = \pi - \gamma/2$ to $\pi + \gamma/2$ and from $\theta = -\gamma/2$ to $+\gamma/2$. Determine the potential and plot the current flow lines.

5. (*10 pts*) A quantum mechanical treatment of the harmonic oscillator results in the Hermite differential equation

$$y'' - 2xy' + \lambda y = 0 \, . \tag{5}$$

(a) Write this equation in standard Sturm-Liouville form.

(b) If the boundary conditions are $y(x) \to 0$ as $x \to \pm\infty$, show that the solutions are orthogonal on the range $(-\infty, +\infty)$, and find the weight function $w(x)$.

(c) Solve the equation to find a series expansion for the Hermite functions. What value of the eigenvalue λ is required for the functions to remain bounded throughout the interval, including $x \to \pm\infty$?

(d) Normalize the solutions by choosing the coefficient of the highest power x^n to be 2^n, and hence determine the first three eigenfunctions.