Name: Solutions - spotting errors ⇒ extra credit

Do NOT panic if you can not finish both problems. Do as much as you can.

Show your work. Results without any derivation do not receive credit. Derive all results starting from the equations below. If you have memorized an equation that results from these four, derive it for full credit. Use this page for scratch work.

\[ \Delta U = Q + W \quad W = -P \, dV \]

\[ U_{\text{thermal}} = \frac{1}{2} N f k T \quad PV = N k T \quad PV^\gamma = \text{constant} \quad (\gamma = 5/3) \]

\[ S = k \ln(\Omega) \quad \frac{1}{T} = \left( \frac{\partial S}{\partial U} \right)_{N,V} \quad \text{ideal gas: } \Omega = f(N)V^N U^{3N/2} \]

\[ \ln(n!) \approx n \ln(n) - n \quad \ln(1 + x) = x - \frac{x^2}{2} + \cdots \]
(100 pts) Given $T_1$, $V_1$ and $V_2$ (express all answers in terms of these three givens), compute: (a) $P_1$, $P_3$, $T_2$ and $T_3$, (b) $W_{12}$, $W_{23}$, $W_{31}$, (c) $Q_{12}$, $Q_{23}$, $Q_{31}$, and (d) $e = 1 - Q_c/Q_h$. 

\[
\begin{align*}
\text{constant pressure} & \quad \Delta V = \frac{3}{2} N k T \Delta T = \frac{3}{2} N k (T_3 - T_2) = -\frac{3}{2} N k T_1 (1 - \frac{V_1}{V_2}) \\
\Rightarrow Q_{32} &= \Delta U - \omega = -\frac{3}{2} N k T_1 (1 - \frac{V_1}{V_2}) \\
\text{constant volume} & \quad W_{31} = 0 \\
\Delta U &= Q = \frac{3}{2} N k \Delta T, \quad \Delta P = N k \Delta T \\
Q_{31} &= \frac{3}{2} N k \Delta T = \frac{3}{2} N k T_1 (1 - \frac{V_1}{V_2}) \\
\text{efficiency} &= 1 - \frac{Q_c}{Q_{12}} = 1 - \frac{\frac{3}{2} N k T_1 (1 - \frac{V_1}{V_2})}{\frac{3}{2} N k T_1 (1 - \frac{V_1}{V_2}) + \frac{3}{2} N k T_1 (1 - \frac{V_1}{V_2})} \\
e &= 1 - \frac{\frac{3}{2} (1 - \frac{V_1}{V_2})}{\ln \left(\frac{V_1}{V_2}\right) + \frac{3}{2} (1 - \frac{V_1}{V_2})} = 1 - \frac{\Delta V}{V_2 \ln \left(1 + \frac{\Delta V}{V_1} + \frac{1}{2} \Delta V\right)}, \quad \Delta V = V_2 - V_1
\end{align*}
\]
Extra credit: Compute the efficiency for an engine with an identical $T_1$, $V_1$ and $V_2$ as in the problem above, but with an adiabatic first step rather than an isothermal first step. Which engine is more efficient?

adiabatic

\[ P_1V_1^\gamma = P_2V_2^\gamma \Rightarrow P_2 = (\frac{V_1}{V_2})^\gamma P_1 \]

\[ P_2V_2 = NkT_2 = P_1V_1(\frac{V_1}{V_2})^\gamma (\frac{V_2}{V_1}) = NkT_1 (\frac{V_1}{V_2})^\gamma \]

\[ \Rightarrow T_2 = (\frac{V_1}{V_2})^\gamma T_1 \]

gas does work + T drops

\[ \Delta U = \frac{W}{V_1} = \frac{3}{2} Nk \Delta T = \frac{3}{2} Nk T_1 (1 - (\frac{V_1}{V_2})^\gamma) \]

\[ Q_{v} = 0 \]

constant pressure

\[ \omega_{23} = -P_2 (V_2 - V_3) = - (\frac{V_1}{V_2})^\gamma P_1 (V_1 - V_2) = - (\frac{V_1}{V_2})^\gamma P_1 V_1 (\frac{V_1}{V_2} - 1) \]

\[ V_{23} = NkT_1 (\frac{V_1}{V_2})^\gamma (1 - \frac{V_1}{V_2}) \]

\[ \Delta U_3 = \frac{3}{2} Nk \Delta T = \frac{3}{2} Nk (\frac{V_1}{V_2})^\gamma T_1 (1 - (\frac{V_1}{V_2})^\gamma T_1) \]

\[ = - \frac{3}{2} Nk T_1 (\frac{V_1}{V_2})^\gamma (1 - \frac{V_1}{V_2}) \]

\[ Q_{23} = \Delta U_3 - \omega_{23} = - \frac{3}{2} Nk T_1 (\frac{V_1}{V_2})^\gamma (1 - \frac{V_1}{V_2}) \]

constant volume

\[ W = 0 \]

\[ \Delta U = Q = \frac{2}{3} Nk \Delta T \]

\[ \Delta P V = Nk \Delta T \]

\[ Q_{31} = \frac{3}{2} Nk (T_1 - T_3) = \frac{3}{2} Nk T_1 (1 - (\frac{V_1}{V_2})^\gamma) \]

\[ \text{Positive, so this is } Q_n \]

\[ e = 1 - \frac{Q}{Q_n} = 1 - \frac{5}{3} \frac{(\frac{V_1}{V_2})^\gamma - 1}{1 - (\frac{V_1}{V_2})^\gamma} = 1 - \frac{5}{3} \frac{(\frac{V_2}{V_1}) - 1}{(\frac{V_1}{V_2}) - 1} \]