Conservation of Energy for a System of N Particles

Here is a much simpler way to derive conservation of energy than the one I tried to use in class.

Let us assume that we have a collection of N particles, located at \( \mathbf{r}_1, \mathbf{r}_2, \ldots, \mathbf{r}_N \), and having masses \( m_1, m_2, \ldots, m_N \). We write the \( \alpha^{th} \) Cartesian component of the position vector of the \( i^{th} \) particle as \( x_{i\alpha} \) (\( \alpha = 1, 2, 3 \)). Thus, the \( i^{th} \) position vector is \( \mathbf{r}_i = (x_{i1}, x_{i2}, x_{i3}) \),

We also assume that the forces acting on the particles can be derived from a potential function. Specifically, we assume that the force on the \( i^{th} \) particle can be written

\[
\mathbf{F}_i = -\nabla_i V(\mathbf{r}_1, \mathbf{r}_2, \ldots, \mathbf{r}_N). \tag{1}
\]

Here \( V \) is a function of the 3N variables \( x_{i\alpha} \) (that is, three Cartesian coordinates for each of the N particles). The operator \( \nabla_i \) is a three-component vector and can be written

\[
\nabla_i = \left( \frac{\partial}{\partial x_{i1}}, \frac{\partial}{\partial x_{i2}}, \frac{\partial}{\partial x_{i3}} \right). \tag{2}
\]

Newton’s second law then takes the form

\[
\frac{d}{dt}(m_i \mathbf{v}_i) = -\nabla_i V(\mathbf{r}_1, \ldots, \mathbf{r}_N). \tag{3}
\]

Now multiply both sides of this equation by \( \mathbf{v}_i \) and sum over \( i \) to get

\[
\sum_{i=1}^{N} m_i \mathbf{v}_i \cdot \frac{d}{dt} \mathbf{v}_i = -\sum_{i=1}^{N} \mathbf{v}_i \cdot \nabla_i V(\mathbf{r}_1, \ldots, \mathbf{r}_N), \tag{4}
\]

where we assume that the \( m_i \)'s are independent of time.

Consider first the left-hand side of eq. (4). We can use the relation

\[
\mathbf{v}_i \cdot \frac{d}{dt} \mathbf{v}_i = \frac{d}{dt} \frac{1}{2} \mathbf{v} \cdot \mathbf{v} \tag{5}
\]
to write the left-hand side as

$$\frac{d}{dt} \sum_{i=1}^{N} \frac{1}{2} m_i v_i^2,$$

(6)

where $v_i^2 = v_i \cdot v_i$.

Now consider the right-hand side of eq. (4). Writing $v_i = dr_i/dt$, we can write the right-hand side as

$$- \sum_{i=1}^{N} \nabla_i V(r_1, ..., r_N) \cdot \frac{dr_i}{dt} = - \sum_{i=1}^{N} \sum_{\alpha=1}^{3} \frac{\partial V}{\partial x_{i\alpha}} \frac{dx_{i\alpha}}{dt} = - \frac{dV}{dt},$$

(7)

where the last equality follows from the chain rule in calculus of several variables. That is, the potential energy $V$ depends on time only implicitly, through the variables $x_{i\alpha}$, i.e., the components of the position vectors.

Setting expressions (6) and (7) equal, we obtain

$$\frac{dT}{dt} = - \frac{dV}{dt},$$

(8)

where

$$T = \sum_{i=1}^{N} \frac{1}{2} m_i v_i^2.$$  

(9)

Or, simplifying, we get

$$\frac{dE}{dt} = 0$$

(10)

where

$$E = T + V$$

(11)

is the total energy. Thus, if the force acting on the $i^{th}$ particle can be obtained as the gradient of a potential, as in eq. (1), the total energy, as defined in eq. (11), is a constant of the motion.