Principle of Least Action

First, introduce generalized coordinates.

If we have $N$ particles in space, there are $N$ position vectors $\mathbf{r}_1, \ldots, \mathbf{r}_N$.

$N$ velocities $\mathbf{v}_1, \ldots, \mathbf{v}_N$

$$= \left( \frac{d\mathbf{r}_1}{dt}, \ldots, \frac{d\mathbf{r}_N}{dt} \right)$$

or $3N$ coordinates (or "degrees of freedom") and $3N$ velocities.

These don't have to be the Cartesian coordinates but could be any other convenient combination, say $q_1, \ldots, q_5$ where $s = 3N$

These are called the generalized coordinates and

$\dot{q}_1, \ldots, \dot{q}_5$ (time derivatives) are called generalized velocities.

If $q_1, \ldots, q_5$, and $\dot{q}_1, \ldots, \dot{q}_5$ are specified at a given time, then it is known experimentally that the subsequent motion of the system is completely determined, i.e., all the accelerations $\ddot{q}_1, \ldots, \ddot{q}_5$ are determined uniquely.
The equations for the accelerations are known as equations of motion.

Integrating the eqs. of motion gives the path of the system.

There are 2s initial conditions on specifying \( q_1, \ldots, q_5, \dot{q}_1, \ldots, \dot{q}_5 \) at time to.

Principle of Least Action (Hamilton's Principle).

Every mechanical system is described by a certain function called a Lagrangian

\[ L(q_1, \ldots, q_5; \dot{q}_1, \ldots, \dot{q}_5; t) = L(q, \dot{q}, t) \]

and the motion of the system is such that a certain condition on L is satisfied.

Let the config. of the system at times \( t_1 \) and \( t_2 \) be defined by \( q^{(1)}(t), q^{(2)}(t) \) or more briefly \( q^{(1)} \) and \( q^{(2)} \).

Then the condition is that the system moves between \( q^{(1)} \) and \( q^{(2)} \) in such a way that

\[ S = \int_{t_1}^{t_2} L(q, \dot{q}, t) \, dt \]

is a minimum.

This function is called the action and the principle is called the principle of least action.

Note: the 2s arbitrary constants which have to be specified are \( q_{10}^{(1)}(t) \), \( q_5^{(1)}(t) \), \( q_1(t_2) \), \( q_5(t_2) \).
Lagrange's equations from the principle of least action

Start by assuming a system with only one degree of freedom. Then

\[ S = \int_{t_1}^{t_2} L(q, \dot{q}, t) \, dt = \min \equiv S_0 \]

where \( q(t_1) = q^{(1)}; \ q(t_2) = q^{(2)} \)

Then let the optimal path be as below:

![Optimal path diagram]

But by Taylor's theorem,

\[ \delta S = \text{difference between } S - S_0 \]

\[ = \int_{t_1}^{t_2} \delta L(q, \dot{q}, t) \, dt \]

\[ \approx \int_{t_1}^{t_2} \frac{\partial L}{\partial q} \delta q(t) \, dt + \int_{t_1}^{t_2} \frac{\partial L}{\partial \dot{q}} \delta \dot{q}(t) \, dt \]
Second term may be written
\[ \int_{t_1}^{t_2} \frac{\partial \mathcal{L}}{\partial \dot{q}(t)} \frac{d}{dt} \delta q(t) \, dt = \left[ \frac{\partial \mathcal{L}}{\partial \dot{q}} \right]_{t_1}^{t_2} - \int_{t_1}^{t_2} \frac{d}{dt} \left( \frac{\partial \mathcal{L}}{\partial \dot{q}} \right) \delta q(t) \, dt \]

But \( \delta q(t_1) = \delta q(t_2) = 0 \) since end points are fixed, hence the first term on the rhs is zero and we get
\[ \delta \mathcal{S} = \int_{t_1}^{t_2} \left[ \frac{\partial \mathcal{L}}{\partial q} - \frac{d}{dt} \left( \frac{\partial \mathcal{L}}{\partial \dot{q}} \right) \right] \delta q(t) \, dt \]

\( = 0 \) if the path is an extremum.

Since this must be true for any \( \delta q(t) \), the coeff. of \( \delta q(t) \) must equal zero, or
\[ \frac{\partial \mathcal{L}}{\partial q} - \frac{d}{dt} \left( \frac{\partial \mathcal{L}}{\partial \dot{q}} \right) = 0 \]

Lagrange's equation

In general, this will be a second order differential eq. in \( q \).
What if we have a Lagrangian which depends on 5 different $q$'s and $\dot{q}$'s.

Then

$$\delta S = \int_{t_1}^{t_2} \delta L \left( q_1(t), \ldots, q_5(t), \dot{q}_1(t), \ldots, \dot{q}_5(t) \right) dt$$

$$= \int_{t_1}^{t_2} \sum_{j=1}^{5} \left( \frac{\partial L}{\partial \dot{q}_j(t)} \delta \dot{q}_j(t) + \frac{\partial L}{\partial q_j(t)} \delta q_j(t) \right) dt$$

But just as in the one-variable case, we can integrate the second terms by parts to get

$$\delta S = \int_{t_1}^{t_2} \sum_{j=1}^{5} \frac{\partial L}{\partial q_j(t)} \delta q_j(t) dt$$

$$+ \sum_{j=1}^{5} \left[ \frac{\partial L}{\partial \dot{q}_j(t)} \right]_{t=t_1}^{t=t_2} \delta q_j(t_1) - \sum_{j=1}^{5} \int_{t_1}^{t_2} \frac{d}{dt} \left( \frac{\partial L}{\partial \dot{q}_j(t)} \right) \delta q_j(t) dt$$

0 because $\delta q_j(t_1) = \delta q_j(t_2) = 0$ (fixed end points)

$$= \sum_{j=1}^{5} \int_{t_1}^{t_2} \left[ \frac{\partial L}{\partial q_j(t)} - \frac{d}{dt} \left( \frac{\partial L}{\partial \dot{q}_j(t)} \right) \right] \delta q_j(t) dt$$

= 0 at extremum

Coeff. of each $\delta q_j(t)$ must vanish independently, so we get
\[
\frac{d}{dt} \left( \frac{\partial L}{\partial \dot{q}_j} \right) - \frac{\partial L}{\partial q_j} = 0, \quad j = 1, \ldots, S.
\]

Lagrange's equations

5 coupled second order differential eqs.

General remark: L is not unique. If we add to L a total time derivative of any function, the Lagrange equations are unchanged. i.e. suppose we replace \( L \) by \( L + \frac{d}{dt} f(q, \dot{q}, t) \)

Then

\[
\delta S = \int_{t_1}^{t_2} \delta L \, \delta(q, \dot{q}, t) \, dt = \int_{t_1}^{t_2} \frac{df}{dt} \, dt + \int_{t_1}^{t_2} \frac{df}{dt} \, dt.
\]

But the second term is just

\[
f(q_1, t_2) - f(q_1, t_1)
\]

independent of path, and so it will not cause change when \( L \) is varied. Hence it has no influence on the Lagrange equations.

E.g. suppose \( f = Aq^2 \)

\[
\frac{df}{dt} = 2Aq \dot{q}
\]

Suppose this term is added to the Lagrangian.

Then

\[
\frac{d}{dt} \left( \frac{\partial L}{\partial \dot{q}_j} \right) - \frac{\partial L}{\partial q_j} = 2A \ddot{q}
\]
\[ \frac{d}{dt} \left( \frac{\partial}{\partial q} (2Aqq) \right) = \frac{d}{dt} (2Aq) = 2Aq \]

Hence \[ \frac{d}{dt} \left( \frac{\partial}{\partial q} (2Aqq) \right) - \frac{\partial}{\partial q} (2Aqq) = 0 \]

so this extra term does not contribute to the equations of motion.

What is the Lagrangian for, say, one particle?

We must first consider the concept of an "inertial frame of reference."

This is a frame in which

One way to define an inertial frame is to say that it is one in which space is homogeneous and isotropic and time is homogeneous.

"Space is homogeneous" means each point in space is equivalent to others.

"Time is homogeneous" means each instant of time is equivalent to every other one.

"Isotropic" means that a body behaves the same way no matter how it is oriented.

Consider a particle in an inertial frame moving freely in such an frame.

Then \( L \) of such a particle cannot depend on \( P \) or on \( t \) (time) or properties would depend on position or time.
Also, it cannot depend on the direction of $\vec{v}$ but only its magnitude, so

$$L = L(v^2)$$

Then the Lagrange's equations are

$$\frac{d}{dt} \left( \frac{\partial L}{\partial \dot{v}_i} \right) = \sqrt{\Sigma} \neq i = 1, 2, 3$$

$\dot{v}_i = i$th Cartesian component of velocity

\[
\frac{\partial L}{\partial v_i} = \text{const.}
\]

but

\[
\frac{dL}{dv_i} = \frac{\partial L}{\partial (v^2)} \frac{\partial v^2}{\partial v_i} = 2v_i \frac{\partial L}{\partial v^2} \dot{v}_i = \text{const.}
\]

This is only possible if $v_i = \text{const.}$

or $\vec{v} = \text{const.}$

So $\vec{v} = \text{const.}$ for a particle free particle in an inertial frame.

(Newton's First Law)

We can obtain $L(v^2)$ from the Galileo Galilei's relativity principle. This principle states that the Laws of Physics have to be the same in two inertial frames of reference, say $K$ and $K'$, such that $K'$ moves with
a constant velocity relative to K.

Let \( r = \text{coord of particle in } K \)
\[ r' = \text{coord of particle in } K' \]

We can see that
\[ r = r' + \vec{v} t' \]
\[ t = t' \]

\( t = t' \): This says that time is absolute
(not true in special relativity)

From the above equations of transformation
\[ \frac{\vec{r}}{\vec{t}} = \frac{d\vec{r}'}{dt} = \frac{d\vec{r}'}{dt'} + \vec{V} \]
\[ = \vec{r}' + \vec{V} \]

Now consider
\( L \) on the primed frame
\[ L(\vec{r}') = L(\vec{t}'^2 - \vec{V}'^2) \]
\[ = L(\vec{t}'^2 - 2\vec{t}' \cdot \vec{V}' + \vec{V}'^2) \]

Suppose \( \vec{V} \) is very small. Then
\[ L(\vec{t}'^2 - 2\vec{t}' \cdot \vec{V}' + \vec{V}'^2) \approx L(\vec{t}'^2 - 2\vec{t}' \cdot \vec{V}') \]
\[ \approx L(\vec{t}'^2) - 2 \vec{t}' \cdot \vec{V}' \frac{dL}{d(\vec{t}'^2)} \]
The second term on the right is a total time derivative only if
\[ \frac{\partial L}{\partial (v^2)} \] is independent of \( v \).

If that is true then
\[ -2 \, \tilde{F} \cdot \tilde{v} \, \frac{\partial L}{\partial (v^2)} = \frac{d}{dt} \left[ -2 \, \tilde{F} \cdot \tilde{v} \, \frac{\partial L}{\partial (v^2)} \right] \]

Therefore, in order for the laws of physics to be the same in all different inertial frames, we must have that \( L \) for a free particle is

\[ L = Av^2 \]

where \( A = \text{some const.} \).

We call \( A = \frac{m}{2} \), \( m = \text{mass of the particle} \).

So
\[ L = \frac{1}{2} m \dot{v}^2 \]

for a free particle.

\( m > 0 \) or otherwise, \( S = \text{maximum on correct path, not a minimum} \).

Lagrange's eq. is
\[ \frac{d}{dt} \left( \frac{\partial L}{\partial \dot{v}_i} \right) = 0 \]

or \( m\dot{v}_i = \text{const.} \).

or \( m\ddot{v} = \text{const.} \) momentum const.

for a free particle.
\[ S = \int_{t_1}^{t_2} \frac{1}{2} m \dot{u}^2 \, dt \leq \text{max. (if } m \text{ is negative) along the actual (straight line) path} \]

Lagrangian for a collection of free particles is

\[ L = \sum_{i=1}^{N} \frac{1}{2} m_i \dot{u}_i^2 \]

Additivity: If total Lagrangian is multiplied by a const., the equations of motion are unchanged.

Note that, for a single particle in Cartesian coordinates,

\[ L = \frac{1}{2} m (\dot{u}_x^2 + \dot{u}_y^2 + \dot{u}_z^2) = \frac{1}{2} m (\ddot{x}^2 + \ddot{y}^2 + \ddot{z}^2) \]

but for other coordinates, the form of L takes a different form.

L for a collection of interacting particles:

\[ L = \sum_{\alpha=1}^{N} \frac{1}{2} m_{\alpha} \dot{u}_{\alpha}^2 - U(\vec{r}_1, \ldots, \vec{r}_N) \]

\( U \) is called the potential energy.
(not necessarily true in special relativity)

This assumes that interactions propagate infinitely fast

Eqs. of motion become

\[ \frac{d}{dt} \left( \frac{\partial L}{\partial \dot{x}^\alpha} \right) = \frac{\partial L}{\partial x^\alpha} \]

Here \( \frac{\partial L}{\partial \dot{x}^\alpha} \) means \( \nabla_{\dot{x}^\alpha} L \)

\( \frac{\partial L}{\partial x^\alpha} \) means \( \nabla_{x^\alpha} L \)

But \( \nabla_{\dot{x}^\alpha} L = m \ddot{x}^\alpha \)

\( \nabla_{x^\alpha} L = -\nabla_{x^\alpha} U \)

So

\[ \frac{d}{dt} \left( \frac{\partial L}{\partial \dot{x}^\alpha} \right) = \frac{\partial L}{\partial x^\alpha} \]

becomes

\[ m \frac{d\dot{x}^\alpha}{dt} = -\nabla_x U \]

"Newton's second law"

\( \nabla_x U \) is the force acting on the \( \alpha \)th particle

How about external potentials (or forces)?

E.g. a single particle in an external field

\[ L = \frac{1}{2} mv^2 - U(\mathbf{r}, t) \]
Which leads to:

\[ m_\ddot{z} = - \nabla_U U \]

Uniform external force \( \mathbf{F} \): \( U = -F \cdot \mathbf{r} \).

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Finally, constraints.

If we neglect friction and also the masses of the constraining elements, the constraints just reduce the number of degrees of freedom.

(These so-called constraints are usually called holonomic constraints.)

Example: Atwood's machine:

\[ L = T - U \]

\[ T = \frac{1}{2} m_1 z_1^2 + \frac{1}{2} m_2 z_2^2 - m_1 g z_1 - m_2 g z_2 \]
But our constraint is that $\dot{z}_1 = -\dot{z}_2$
and also $z_1 + z_2 = -(l_1 + l_2) = \text{const.}$

(if we measure the change $z$ from the center of
the pulley wheel).

Thus we have (writing

$$L = \frac{1}{2} (m_1 + m_2) \dot{z}_1^2 - m_1 g z_1 - m_2 g (-z_1 - \frac{e}{2})$$

$$= \frac{1}{2} (m_1 + m_2) \dot{z}_1^2 + (m_2 - m_1) g z_1 + m_2 g l$$

$$\frac{d}{dt} \left( \frac{\partial L}{\partial \dot{z}_1} \right) - \frac{\partial L}{\partial z_1} = (m_1 + m_2) \ddot{z}_1$$

$$- (m_2 - m_1) g \dot{z}_1 = 0$$

or

$$\ddot{z}_1 = \frac{m_2 - m_1}{m_2 + m_1} g$$

in agreement with elementary arguments

(but we don't get out the tension in the rope
from this formulation).

Find the Lagrangian of a coplanar double
pendulum placed in a uniform
gravitational field.
Generalized coordinates are \( \phi_1 \) and \( \phi_2 \)

\[
U = -m_1 g l_1 \cos \phi_1 - m_2 g (l_2 \cos \phi_2 + l_1 \cos \phi_1)
\]

\[
T_1 = \frac{1}{2} m (x_1^2 + y_1^2)
\]

\[
x_1 = l_1 \sin \phi_1 \quad \dot{x}_1 = l_1 \cos \phi_1 \dot{\phi}_1
\]

\[
y_1 = l_1 \cos \phi_1 \quad \dot{y}_1 = -l_1 \sin \phi_1 \dot{\phi}_1
\]

\[
\ddot{x}_1^2 + \ddot{y}_1^2 = l_1^2 (\sin^2 \phi_1 + \cos^2 \phi_1) \dot{\phi}_1^2 = l_1^2 \dot{\phi}_1^2
\]

\[
T_2 = \frac{1}{2} m (x_2^2 + y_2^2)
\]

\[
x_2 = l_1 \sin \phi_1 + l_2 \sin \phi_2 \quad \dot{x}_2 = l_1 \cos \phi_1 + l_2 \cos \phi_2 \dot{\phi}_2
\]

\[
y_2 = l_1 \cos \phi_1 + l_2 \cos \phi_2 \quad \dot{y}_2 = l_1 \sin \phi_1 \dot{\phi}_1 + l_2 \sin \phi_2 \dot{\phi}_2
\]

\[
\ddot{x}_2^2 + \ddot{y}_2^2 = l_1^2 \dot{\phi}_1^2 + l_2^2 \dot{\phi}_2^2 + 2 l_1 l_2 \left( \cos \phi_1 \cos \phi_2 + \sin \phi_1 \sin \phi_2 \right) \dot{\phi}_1 \dot{\phi}_2
\]

So finally,

\[
L = \frac{1}{2} \left( m_1 + m_2 \right) l_1^2 \dot{\phi}_1^2 + \frac{1}{2} m_2 l_2^2 \dot{\phi}_2^2 + 2 m_2 l_1 l_2 \dot{\phi}_1 \dot{\phi}_2 \cos (\phi_1 - \phi_2)
\]

\[
+ \left( m_1 - m_2 \right) g l_1 \cos \phi_1 + m_2 g l_2 \cos \phi_2
\]