Need to connect scattering cross-sections in the two frames. We will not discuss in this course; see GPS, sec. (3, 11) if interested.

Kinematics of Rigid Body Motion (GPS, Chapter 4)

Rigid body: a collection of \( N \) particles \((N \text{ maybe very large})\) such that the distance \( r_{ij} \) between any two is fixed \((ab)\).

How many indep coordinates needed to specify position and orientation of a rigid body?

It turns out: 6.

Why?

Well, we have \( N \) particles \( \Rightarrow 3N \) coords.

constraints: \( 2N(N-1) \) distances fixed \( \Rightarrow 3N \) for \( N \text{ large} \)

so most of constraints are not independent.

Actually, if you fix 3 the coords of 3 mass points on the rigid body, the rest of the \( N \) mass points have fixed position because they have fixed distances to those three \((\text{non-collinear})\) points.

Hence

\[ \text{Specifying the} \]

\[ \text{dists.} \quad r_{14}, r_{12}, r_{42}, r_{43} \]

\[ \text{tells you where particle 4 is, because mass that is three coordinates} \]
So # of degrees of freedom = 9 (# of words of 3 particles) - 3 (three specified distances) = 6.

How to choose the 6 coordinates?

First, locate a set of Cartesian "body" coordinates outside the body which are fixed in the rigid body.

E.g.,

Second set of three words needed to specify orientation of (\(x', y', z'\)) relative to lab set coord. (\(x, y, z\)) (need three words)

Many ways to do this.

E.g. consider unit vectors \(\hat{i}, \hat{j}, \hat{k}\) of unprimed and \(\hat{i}', \hat{j}', \hat{k}'\) of primed coordinate axes

Then there are 9 direction cosines, e.g.

\[
\cos \theta_{12} = \frac{\hat{i} \cdot \hat{i}'}{\hat{i} \cdot \hat{i}'} = \cos \theta_{12}
\]
For example, we write \( \mathbf{x} \cdot \mathbf{y} = \cos \theta_{12} \)

where \( \theta_{12} = \) angle between \( \mathbf{x} \) and \( \mathbf{y} \).

We also denote \( \cos \theta_{ij} = a_{ij} \) and in general

\[
    \cos \theta_{ij} = \hat{x}_i \cdot \hat{x}_j = a_{ij}
\]

where I have written \( \hat{x}_i, \hat{y}_i, \hat{z}_i \)

the unit vectors along \( x, y, \) and \( z \) axes, as

\[
    \hat{x}_1, \hat{x}_2, \hat{x}_3
\]

and \( \hat{x}', \hat{y}', \hat{z}' \) as \( \hat{x}_1', \hat{x}_2', \hat{x}_3' \)

Then we have

\[
    \begin{align*}
    \hat{x}_1' &= a_{11} \hat{x}_1 + a_{12} \hat{x}_2 + a_{13} \hat{x}_3 \\
    \hat{x}_2' &= a_{21} \hat{x}_1 + a_{22} \hat{x}_2 + a_{23} \hat{x}_3 \\
    \hat{x}_3' &= a_{31} \hat{x}_1 + a_{32} \hat{x}_2 + a_{33} \hat{x}_3
    \end{align*}
\]

or more compactly

\[
    \hat{x}_i' = \sum_{j=1}^{3} a_{ij} \hat{x}_j
\]

Note that we could express a vector \( \mathbf{r} \) either as

\[
    \mathbf{r} = \hat{x}_1 \hat{x}_1 + \hat{x}_2 \hat{x}_2 + \hat{x}_3 \hat{x}_3
\]

or as

\[
    \mathbf{r} = \hat{x}_1 \hat{x}_1' + \hat{x}_2 \hat{x}_2' + \hat{x}_3 \hat{x}_3'
\]

i.e. either as \( (x_1, x_2, x_3) \) or as \( (x_1', x_2', x_3') \)

Now we can take the second of these equations and write

\[
    \mathbf{r} = \sum_{i=1}^{3} x_i \hat{x}_i
\]

or

\[
    \mathbf{r} = \sum_{i=1}^{3} x_i \hat{x}_i' = \sum_{i=1}^{3} \sum_{j=1}^{3} a_{ij} x_i \hat{x}_j = \sum_{i=1}^{3} \sum_{j=1}^{3} a_{ij} x_i \hat{x}_j
\]
\[ \sqrt{x'_i} = \sum_{i=1}^{3} a_{ij} \cdot x'_j \]

which provides a relation between components of the same vector expressed in two different orthogonal coordinate systems.

Comment: my notation is (slightly) different from GPS.

Now: 9 direction cosines $a_{ij}$ but there should be only 3 distinct angles. So there should be 6 relations between these cosines. These relations are:

\[ \hat{x}'_i \cdot \hat{x}'_j = \hat{x}'_2 \cdot \hat{x}'_2 = \hat{x}'_3 \cdot \hat{x}'_3 = 1 \]
\[ \hat{x}'_1 \cdot \hat{x}'_2 = \hat{x}'_2 \cdot \hat{x}'_3 = \hat{x}'_3 \cdot \hat{x}'_1 = 0 \]

which can be expressed as

\[ \hat{x}'_i \cdot \hat{x}'_j = \delta_{ij} \]

where \[ \delta_{ij} = \begin{cases} 1 & i = j \\ 0 & i \neq j \end{cases} \]

(Kronecker delta function)

These are the six relations, so there really are only three added independent angular parameters for a rigid body. In terms of the $a_{ij}$ matrix

\[ \hat{x}'_i = \sum_{k=1}^{3} a_{ik} \hat{x}'_k \]
\[ \hat{x}'_j = \sum_{l=1}^{3} a_{jl} \hat{x}'_l \]
So
\[ \nabla_i \cdot \nabla_j = \sum_{k=1}^{3} \sum_{l=1}^{3} a_{ik} a_{jl} \nabla_k \cdot \nabla_l \]
\[ = \sum_{k} a_{ik} a_{jk} = \delta_{ij} \quad i, j = 1, 2, 3 \]

This is a set of 6 equations on the direction cosine matrix \( a_{ij} \).

If you write the quantities \( a_{ij} \) as a matrix \( \mathbf{A} \)

\[
\mathbf{A} = \begin{pmatrix}
  a_{11} & a_{12} & a_{13} \\
  a_{21} & a_{22} & a_{23} \\
  a_{31} & a_{32} & a_{33}
\end{pmatrix}
\]

we see that the above equation just says

\[ \mathbf{A} \mathbf{A} = \mathbf{I} \Rightarrow (\mathbf{A}^T \mathbf{A})_{ij} = \delta_{ij} = \sum_{k} a_{ik} a_{jk} \]

where \( \mathbf{A}^T \) is the transpose of \( \mathbf{A} \):

\[
(\mathbf{A}^T)_{ij} = (\mathbf{A})_{ji}
\]

Then the equation at the top of p. 112 can be written

\[
\mathbf{X} = \mathbf{A} \mathbf{X}'
\]

where

\[
\mathbf{X} = \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} \quad \text{and} \quad \mathbf{X}' = \begin{pmatrix} x'_1 \\ x'_2 \\ x'_3 \end{pmatrix}
\]

This follows because

\[
x_j = (\mathbf{A})_{ji} \sum_{i=1}^{3} (\mathbf{A}^T)_{ij} x'_i
\]
or \( x_j = \frac{3}{2} a_{ij} x_i \)

But we can also multiply both sides by \( a \):

\[ ax = ax' = x' \]

so \[ x' = ax \]

\( x' = \) column vector
\( x = \) column vector

---

Orthogonal transformations

Side discussion: orthogonal transformations

Let us consider a set of transformations

\[ x'_i = \sum_{j=1}^{3} a_{ij} x_j \]

where the length is preserved, i.e.

\[ \sum_{i=1}^{3} x'_i x'_i = \sum_{i=1}^{3} x_i x_i \]

For an example of this, we expressing the same vector in two different Cartesian coordinate systems

Since \[ \sum_{j=1}^{3} a_{ij} x'_j = \sum_{j=1}^{3} a_{ij} x_j \]

and we may write

\[ \sum_{i=1}^{3} x'_i x'_i = \sum_{i=1}^{3} \sum_{j=1}^{3} a_{ij} x_j a_{ik} x_k = \sum_{i=1}^{3} \sum_{j=1}^{3} \sum_{k=1}^{3} a_{ij} a_{ik} x_j x_k \]
\[
= \sum_{j=1}^{3} \sum_{k=1}^{3} x^i_j x_k^j \sum_{i=1}^{3} a_{ij} a_{ik} \\
\text{Since this relation must hold for any } x_i, x_j, \text{ we must have}
\sum_{i=1}^{3} a_{ij} a_{ik} = \delta_{jk}
\]

Then rhs becomes
\[
\sum_{j=1}^{3} \sum_{k=1}^{3} x^j_k \delta_{jk} = \sum_{j=1}^{3} x^j_j = \sum_{i=1}^{3} x^i_i
\]

So, if this linear transformation is to preserve length, we must have
\[
\sum_{i=1}^{3} a_{ij} a_{ik} = \delta_{jk} = \sum_{i=1}^{3} \alpha_i (\bar{\alpha})_i a_{ik} = (\bar{\alpha} \alpha)_j
\]

or, as matrices

\[
I = \bar{\alpha} \alpha 
\]

(1)

Comment: If \( \bar{\alpha} \alpha = I \), then \( \bar{\alpha} \bar{\alpha} = I \) also.

Proof: Right multiply the above by \( \bar{\alpha} \). Then since \( \bar{\alpha} \alpha^{-1} = I \), we get \( \bar{\alpha} = \alpha^{-1} \).

Now left-multiply by \( \alpha \) and we get
\[
\bar{\alpha} \alpha = \alpha \alpha^{-1} = I \quad \text{QED}
\]

Another method: Start from (1), left-multiply by \( \alpha \), right-multiply by \( \bar{\alpha} \).
Then we get 
\[ a \hat{a} a \hat{a} = a \hat{a} I \hat{a} = a \hat{a} \]

or 
\[ (a \hat{a})^2 = (a \hat{a}) = a \hat{a} = I. \quad \text{QED} \]

Thus, the matrix \( a \) which describes a length-preserving linear transformation (such as a rotation) also obeys \( a \hat{a} = \hat{a} a = I \). It is called an orthogonal matrix. This is called an orthogonal transformation.

Notes: 
(i) All elements of \( a \) are real.

(ii) Use "Einstein convention": repeated indices summed over. 
\[ (AB)_{ij} = \sum_k A_{ik} B_{kj} = A_{ik} B_{kj} \]

E.g. 
\[ a \hat{a} = \sum_i A_i A_i = A_i A_i \]
\[ A \hat{A} = \sum_i A_i^2 = A_i A_i \]

\( n \times n \) (will not use this very consistently)

The \( a_{ij} \)'s are matrix elements of the transformation.

Simple example: rotation in a plane
In the unprimed system \( \vec{p} = x_1 \hat{x}_1 + y_1 \hat{y}_1 + z_1 \hat{z}_1 \)

and in the primed system \( \vec{p} = x_1' \hat{x}_1' + y_1' \hat{y}_1' + z_1' \hat{z}_1' \)

where
\[
\begin{pmatrix}
  x_1' \\
  x_2' \\
  x_3'
\end{pmatrix} =
\begin{pmatrix}
  \cos \phi & \sin \phi & 0 \\
  -\sin \phi & \cos \phi & 0 \\
  0 & 0 & 1
\end{pmatrix}
\begin{pmatrix}
  x_1 \\
  x_2 \\
  x_3
\end{pmatrix}
\]

This matrix is the same one which relates the unit vectors:
\[
\begin{pmatrix}
  \hat{x}_1' \\
  \hat{x}_2' \\
  \hat{x}_3'
\end{pmatrix} =
\begin{pmatrix}
  \cos \phi & \sin \phi & 0 \\
  -\sin \phi & \cos \phi & 0 \\
  0 & 0 & 1
\end{pmatrix}\begin{pmatrix}
  \hat{x}_1 \\
  \hat{y}_1 \\
  \hat{z}_3
\end{pmatrix}
\]

( ) is the matrix \( \hat{\hat{\alpha}} \) for this problem.

By inspection, \( \hat{\hat{\alpha}} \hat{\hat{\alpha}} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \)
\[
\begin{pmatrix}
\cos \phi & \sin \phi & 0 \\
-\sin \phi & \cos \phi & 0 \\
0 & 0 & 1
\end{pmatrix}
\begin{pmatrix}
\cos \phi & -\sin \phi & 0 \\
\sin \phi & \cos \phi & 0 \\
0 & 0 & 1
\end{pmatrix}
\]

The matrix \( A \) of the transformation is

\[
A = \begin{pmatrix}
\cos \phi & \sin \phi & 0 \\
-\sin \phi & \cos \phi & 0 \\
0 & 0 & 1
\end{pmatrix}
\]

Instead of rotating coordinate system
As written, the equation \( \vec{F}' = A \vec{F} \) means
\( \vec{F}' \) and \( \vec{F} \) are equal vectors expressed in two coprime coordinate systems rotated with respect to one another. Could also say that \( A \) describes notation of the vector in an unchanged coordinate system.

If coprime system was rotated \( \phi \) counterclockwise
then in the second case, the vector would be rotated clockwise.

First case: "passive" transformation
Second case: "active" transformation

A few formal properties of transformation matrix:
Let $\mathbf{x}' = \mathbf{Bx}$

and $\mathbf{x}'' = \mathbf{AX}'$

Then $\mathbf{x}'' = \mathbf{ABx} = \mathbf{Cx}$

So $\mathbf{C} = \mathbf{AB}$

and $c_{ij} = \sum_{k=1}^{3} a_{ik} b_{kj}$

Theorem: If $\mathbf{A}$ and $\mathbf{B}$ are orthogonal, then so is $\mathbf{C}$ (exercise)

Note also: $\mathbf{AB} \neq \mathbf{BA}$ in general

So successive notations do not have different effects depending on their order.

However, $\mathbf{A(BC)} = (\mathbf{AB})\mathbf{C} = \mathbf{ABC}$
(matrices multiplication is associative)

so $\mathbf{x}''' = \mathbf{ABCx} = (\mathbf{AB})\mathbf{Cx} = (\mathbf{AB})\mathbf{x}'$

$= \mathbf{A(BC)x} = \mathbf{Ax}''$. $\mathbf{x}'$, $\mathbf{x}''$, $\mathbf{x}'''$ column (3) - vectors

The matrix $\mathbf{A}^{-1}$ is that matrix which, when it is applied multiplies $\mathbf{A}$, gives $\mathbf{I}$, the $3 \times 3$ unit matrix.

Can show in general that $\mathbf{AA}^{-1} = \mathbf{A}^{-1}\mathbf{A}$
Proof:
Let $\tilde{A}^t A = I$

Then $A(\tilde{A}^t A)^{-1} = AIA^{-1} = AA^{-1} = (\tilde{A}^t)^{-1}(A\tilde{A}^{-1})$ since matrix multiplication is associative

So $(AA^{-1})^{-1} = (\tilde{A}^t)^{-1}$

Right multiply by $(\tilde{A}^t)^{-1}$

$AA^{-1} = (\tilde{A}^t)^{-1} = I$

since any matrix times its inverse $= I$

So $A^{-1} = I$ and $A$ and $A^{-1}$ commute. Q.E.D

A square matrix $A$ such that $A = \tilde{A}$ is called symmetric.

If $A = -\tilde{A}$ then $A$ is antisymmetric.

Suppose $G = AF$ where $A$ is some square matrix and $G$ and $F$ are column vectors in some coordinate system.

Suppose we go into a new coordinate system in which $G' = BG$ and $F' = BF$.

Then we have $G' = BAF = BAB^{-1}BF = (B\tilde{A}B^t)F'$

$= A'F'$ where $A' = BAB^{-1}$

So the "operator" $A$ in the unprimed coordinate system becomes $BAB^{-1}$ in the primed coordinate system.
The relation \( A' = B A B^{-1} \) is called a similarity transformation.

**Determinants:**
write \( \det A = |A| \)

**Properties:**
\( |A B| = |A| |B| \)
\( |A'| = |A| \)

**Proof:**
so \( |A A A| = 1 \)

because determinants are unaffected by interchanging rows and columns, as one does in forming the transpose.

Also for

For a rotation, \( \hat{A} \hat{A} = I \)

so \( |\hat{A} A| = |I| = 1 = |\hat{A}| |A| = |A|^2 \)

so \( |A| = \pm 1 \)

+1 "proper rotation" (see below)

-1 "improper rotation"

Determinant is invariant under a similarity transform.

**Proof:** Consider a similarity transform of \( A \):

\[ A' = B A B^{-1} \]

\[ |A'| = |B| |A| |B^{-1}| = |B| |B^{-1}| |A| \]