Spin Angular Momentum

Up to now, we have considered an electron (and other particles) to be point particles, fully described by three degrees of freedom \((x, y, z)\). Wave function is expressible as \(\psi(x, y, z)\) which satisfies a certain Schrödinger equation. This theory gives certain very precise results, e.g. energy levels of hydrogen atom.

However, it turns out that these three degrees of freedom are not enough to describe electron — also need another observable, the electron spin. (And in general, most "elementary particles" have a spin degree of freedom.)

For electrons, spin emerges from relativistic Dirac equation. However, spin was discovered, historically, before Dirac equation. Simple theory due to Pauli allows spin to be incorporated into the Non-Relativistic QM. (Valid when \(v/c \ll 1\).)
Experimental evidence for spin
(mostly electrons)

I. Fine structure of spectral lines

\[ 3s, 3p, 3d \]
\[ 2s, 2p \]

Hydrogen

I. "Anomalous Zeeman effect."

When H atom placed in a magnetic field, the lines (i.e. the fine structure) split further.

This further splitting due to fact that spin angular momentum \( \frac{1}{2} \) has associated with it a "magnetic moment" \( m_S \).

III. Existence of half-integer angular momentum

(Stern-Gerlach experiment).

Beam of silver atoms is split in two by a magnetic field.

Pauli theory of spin: Postulates

A particle has in addition to orbital orbital
degrees of freedom (\( R \) and \( \vec{p} \)) has spin
degrees of freedom
The spin variables satisfy the following postulates:

(i). Spin operator $\mathbf{S}$ is an angular momentum

Thus $\mathbf{S} = (S_x, S_y, S_z)$

$[S_x, S_y] = i\hbar S_z$ and cyclic

(ii). Spin operators operate in a "spin space" $\mathcal{E}_S$

In this space $S^2$ and $S_z$ form a complete set of commuting observables.

Thus a spin state is specified by $|s m_s\rangle$ such that

$$ S^2 |s m_s\rangle = s(s+1) \hbar^2 |s m_s\rangle $$

$$ S_z |s m_s\rangle = \hbar m |s m_s\rangle $$

A given particle has a unique value of $S$.

Eigenvalue of $S^2$ is of finite dimension $2s + 1$.

All states are eigenstates of $S^2$ with same eigenvalue $s(s+1)\hbar^2$.

(iii). State space $\mathcal{E}$ of the particle is the tensor product of $\mathcal{E}_r$ and $\mathcal{E}_S$

$$ \mathcal{E} = \mathcal{E}_r \otimes \mathcal{E}_S $$

$\Rightarrow$ all spin observables commute with all orbital (spatial) observables.
Thus, unless $S = 0$, need to specify a ket both spin variable state and orbital state of a particle. Every particle state is a linear combination of states, each of which is a direct product of a ket $|\psi\rangle$ of $E_x$ and one of $E_x$ (more later).

(iv). The electron is a state of spin $\frac{1}{2}$ and it has an intrinsic angular momentum

$$\vec{M} = 2 \frac{m_B}{h} \vec{S}$$

where $m_B = \frac{e \hbar}{2me} \quad \text{(in SI units)}$

"Bohr magneton"

Comments:

(i). Proton and neutron are also spin $-\frac{1}{2}$ particles but with different gyromagnetic ratio than electron

(ii). "Classical" picture

$\bigcirc$ electron as spinning "sphere"

Not really valid: would need six numbers to specify sphere: 3 position, 3 Euler angles.
Hence $S^2$ has no classical analog.

**Special properties of angular momentum $\frac{1}{2}$**

Eigenstate has a basis of two:

\[ |S=\frac{1}{2}, m_s=\frac{1}{2}\rangle \equiv |+\rangle \]
\[ |S=\frac{1}{2}, m_s=-\frac{1}{2}\rangle \equiv |-\rangle \]

\[ S^2 |\pm\rangle = \frac{3}{4} \hbar^2 |\pm\rangle = S(S+1)\hbar^2 |\pm\rangle \]

\[ S_z |\pm\rangle = \pm \frac{1}{2} \hbar |\pm\rangle \]

\[ \langle +1 | +\rangle = \langle -1 | -\rangle = 1 \]
\[ \langle +1 | -\rangle = \langle -1 | +\rangle = 0 \]
\[ 1 + \langle +1 | +1 \rangle + \langle -1 | -1 \rangle = 1 \]

An arbitrary spin state can be written

\[ |\Psi\rangle = c_+ |+\rangle + c_- |-\rangle \]

where $c_+$, $c_-$ are complex constants.

Since

\[ S^2 |\Psi\rangle = c_+ S^2 |+\rangle + c_- S^2 |-\rangle \]

\[ = \frac{3}{4} \hbar^2 (c_+ |+\rangle + c_- |-\rangle) \]

\[ = \frac{3}{4} \hbar^2 |\Psi\rangle \quad \Rightarrow \quad S^2 = \frac{3}{4} \hbar^2 \]
$^{3}S_{0}$ is an angular momentum, so it satisfies all the properties of an angular momentum.

Thus \[ S_{x, y, z} = S_{x, y, z} \pm iS_{y, x, z} \]

\[ S^{+} \mid + \rangle = \frac{\hbar}{2} \mid + \rangle \]
\[ S^{-} \mid - \rangle = \frac{\hbar}{2} \mid - \rangle \]

Any operator in the $2 \times 2$ \[ \mid + \rangle, \mid - \rangle \] basis can be represented as a $2 \times 2$ matrix.

In particular, we have can show, using above relations, that

\[ \begin{pmatrix} S_{x} \
 S_{y} \end{pmatrix} = \frac{\hbar}{2} \begin{pmatrix} \sigma_{x} & \sigma_{y} \
 \sigma_{y} & \sigma_{x} \end{pmatrix} \]

where \[ \sigma_{x} = \begin{pmatrix} 0 & 1 \
 1 & 0 \end{pmatrix}, \quad \sigma_{y} = \begin{pmatrix} 0 & -i \
 i & 0 \end{pmatrix} \]
\[ \sigma_{z} = \begin{pmatrix} 1 & 0 \
 0 & -1 \end{pmatrix} \]

the famous Pauli matrices.

Proof: E.g.

\[ \begin{pmatrix} S_{x} \
 S_{y} \end{pmatrix} = S^{+} + S^{-} \]
\[ \frac{\hbar}{2} \]
Thus \[ S_x |\uparrow\rangle = \frac{1}{2} (S_+ + S_-) |\uparrow\rangle \]
\[ = \frac{\hbar}{2} |\uparrow\rangle \]
\[ S_x |\downarrow\rangle = \frac{1}{2} (S_+ + S_-) |\downarrow\rangle = \frac{\hbar}{2} |\downarrow\rangle \]
So \[ S_x |\uparrow\rangle = S_x (\frac{\hbar}{2} |\uparrow\rangle + \frac{\hbar}{2} |\downarrow\rangle) \]
\[ = \frac{\hbar}{2} |\uparrow\rangle + \frac{\hbar}{2} |\downarrow\rangle \]
which we can represent as follows. Write 
\[ |\chi\rangle \]
as
\[ \begin{pmatrix} c_+ \\ c_- \end{pmatrix} \]
Then \[ S_x \begin{pmatrix} c_+ \\ c_- \end{pmatrix} = \frac{\hbar}{2} \begin{pmatrix} c_- \\ c_+ \end{pmatrix} \]
which implies that \[ S_x = \frac{\hbar}{2} \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} \]
Properties of Pauli matrices:
\[ \sigma_x^2 \sigma_x = \sigma_y^2 \sigma_x = \sigma_z^2 \sigma_x = 1 \]
\[ \{ \sigma_x, \sigma_y \} = 0 \]
where \{ \sigma_+ \}
\[ \text{denotes the anticommutator} \]
\[ [\sigma_x, \sigma_y] = 2i \sigma_z \text{ and cyclic} \]
\[ \sigma_x \sigma_y = \sigma_y \sigma_x \text{ and cyclic} \]
\[ \text{Tr } \sigma_x = \text{Tr } \sigma_y = \text{Tr } \sigma_z = 0; \text{ det } \sigma_x = \text{ det } \sigma_y = \text{ det } \sigma_z = -1 \]
Any 2x2 matrix can be written as
\[ C_1 I + C_2 \sigma_x + C_3 \sigma_y + C_4 \sigma_z. \]
Also, for any two arbitrary vectors \( \vec{A} \) and \( \vec{B} \)
whose components commute with the components of spin,
\[(\vec{\sigma} \cdot \vec{A})(\vec{\sigma} \cdot \vec{B}) = \vec{A} \cdot \vec{B} + i \vec{\sigma} \cdot (\vec{A} \times \vec{B})\]

Observables and state vectors of an electron,
including both space and spin
degrees of freedom.
"Spinor" Wave Functions

If we just had the three coordinates x, y, z, then the prob. distribution would be described by a wave function of the form \( \psi(x, y, z, t) \). Conversely, if we just had spin degrees of freedom, we could write the state as

\[
|\psi\rangle = c_+ |\uparrow\rangle + c_- |\downarrow\rangle
\]

where, for normalization, \( |c_+|^2 + |c_-|^2 = 1 \).

If we have a spin particle with

we could also write this state as

\[
\begin{pmatrix}
  c_+ \\
  c_-
\end{pmatrix}
\]

If we have both space and spin degrees of freedom, the wave function (for a spin-\(1/2\) particle) will generally be a spinor with two components, of the form

\[
\begin{pmatrix}
  \psi_+(x, t) \\
  \psi_-(x, t)
\end{pmatrix} = \psi_+(x, t) \chi_+ + \psi_-(x, t) \chi_-
\]

where \( \chi_+ = \begin{pmatrix} 1 \\ 0 \end{pmatrix} \) and \( \chi_- = \begin{pmatrix} 0 \\ 1 \end{pmatrix} \).

Prob. that particle has spin up at time + and is located in volume element \( d^3x \) near \( \vec{x} \) is thus \( |\psi_+(x, t)|^2 \).
Normalization is

\[ \int \left[ |\psi_+ (x,t)|^2 + |\psi_- (x,t)|^2 \right] \, d^3x = 1. \]

\[ = \int d^3x \left( \psi_+^* (x,t) \cdot \psi_+ (x,t) + \psi_-^* (x,t) \cdot \psi_- (x,t) \right) \]

Very often, space and part of the wave function is the same for spin up and spin down parts. In this case, we can write

\[
\begin{pmatrix}
\psi_+ (x,t) \\
\psi_- (x,t)
\end{pmatrix} = \begin{pmatrix}
\psi (x,t) \\
\psi (x,t)
\end{pmatrix} \begin{pmatrix} c_+ (t) \\
c_- (t) \end{pmatrix},
\]

i.e. space and spin parts of wave functions are independent.

Representation as a ket:

\[ |\psi (x,t) \rangle = \text{wave function which depends on state vector in } \text{ket form which depends on a discrete index } s = \pm \text{ and a continuous index} \]

Expectation operators in the direct product space.

There are those which act only on \( E_r = \text{spatial} \)}
variables, e.g. position $\vec{r}$, momentum $\vec{p}$, orbital angular momentum $\vec{L} = \vec{r} \times \vec{p}$.

Expected value of position of particle with a spinor wave function is

$$\int d^3\vec{r} \psi^*(\vec{r}, t) \psi(\vec{r}, t) + \psi^*(\vec{r}, t) \psi(-\vec{r}, t)$$

So in spinor representation the position operator is a matrix

$$\begin{pmatrix} \vec{r} & 0 \\ 0 & \vec{r} \end{pmatrix}$$

and

$$\langle \vec{r} \rangle = \int d^3\vec{r} \left( \psi^*(\vec{r}, t) \psi(\vec{r}, t) \right) \begin{pmatrix} \vec{r} & 0 \\ 0 & \vec{r} \end{pmatrix}$$

Similarly the momentum operator is

$$\begin{pmatrix} -i\hbar \vec{\nabla} & 0 \\ 0 & i\hbar \vec{\nabla} \end{pmatrix}$$

There are also some operators which operate only in spin space, e.g. $S_x$, $S_y$, $S_z$.

$S_x$ would be represented as

$$\frac{i\hbar}{2} \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$$

Finally, a mixed operator:
\[
\vec{L} \cdot \vec{S} = L_x S_x + L_y S_y + L_z S_z
\]

Consider just \( L_z S_z \)

\[
L_z = -i\hbar \frac{\partial}{\partial \phi} \quad \text{(acts only on spatial coordo, i.e., \( \phi \) in \( \mathbb{R} \))}
\]

\[
\hat{S}_z = \frac{\hbar}{2} \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}
\]

Then

\[
L_z S_z = -i\hbar \frac{\hbar^2}{2} \begin{pmatrix} \frac{\partial}{\partial \phi} & 0 \\ 0 & -\frac{\partial}{\partial \phi} \end{pmatrix}.
\]

Probability that an electron has angular \( \frac{\hbar}{2} \) spin with respect to the spin \( z \) axis:

\[
P_+ = \int \left| \psi_+ (\vec{r}) \right|^2 d^3r
\]

assuming

\[
\int \left[ \left| \psi_+ (\vec{r}) \right|^2 + \left| \psi_- (\vec{r}) \right|^2 \right] d^3r = 1
\]
Spin dynamics:

For the moment, we consider an electron with only spin degrees of freedom, not orbital degrees of freedom. Then the Schrödinger equation is

$$H|\psi\rangle = \frac{i}{\hbar} \frac{\partial}{\partial t} |\psi\rangle$$

where $|\psi\rangle$ can be represented as a two-component column vector, i.e.

$$|\psi\rangle = \begin{pmatrix} c_+ \\ c_- \end{pmatrix}$$

That is $|\psi\rangle = c_+ |+\rangle + c_- |-_\rangle$

with $|c_+|^2 + |c_-|^2 = 1$

So $|\psi\rangle$ can be represented as

$$|\psi\rangle = \begin{pmatrix} c_+ \\ c_- \end{pmatrix}$$

$H$ will be an operator, which in spin space will be, in general, a 2x2 matrix.

Now, what is $H$? Typically, as I said, as I said,

As I said, an electron has a magnetic moment $\mu$, which is proportional to $S$. 

What is the Hamiltonian?

Well, according to classical electrodynamics, the interaction energy of a magnetic dipole moment with an external magnetic field \( \hat{B} \) is

\[
W = -\hat{\mu} \cdot \hat{B}.
\]

A quick recollection of what this means leads to classically:

\[
\hat{B} = B \hat{z}
\]

This leads to a torque

\[
\tau = \frac{\partial W}{\partial \theta} \hat{x} = -\mu B \sin \theta \hat{x}
\]

and where \( \hat{x} \) is in the direction \( \hat{x} \times \hat{B} \) perpendicular to \( \hat{\mu} \) and \( \hat{B} \).

Or, including the vector perpendicular character of \( \hat{\mu} \) and \( \hat{B} \), correctly, we get

\[
\tau = \hat{\mu} \times \hat{B}
\]
We also know, from classical mechanics, that
\[ \vec{N} = \frac{d\vec{L}}{dt} \]

so
\[ \frac{d\vec{L}}{dt} = \vec{\mu} \times \vec{B} \] (classically)

Finally, suppose the system with the magnetic moment is a current loop. Then
\[ \vec{\mu} = \frac{IA}{c} \hat{e}_\perp \]
where \( I = \text{current}, \ A = \text{area}, \ \hat{e}_\perp \text{ is a unit vector} \) to current loop.

Also, the current is due to each electron is
\[ \vec{J} + \frac{q}{m} \omega \vec{r} = -le \]
whereas the angular momentum is \( NmvR \)

Suppose we have

The current in the loop is
\[ I = nqe \omega \]
where \( n = \text{# of electrons per unit length} \)

or \( I = \frac{Nqe}{2\pi R} \)

\[ \frac{IA}{c} = \frac{Nq\omega R^2}{2\pi R} = \frac{Nq\omega R}{2mc} = \frac{1}{2mc} \frac{dL}{dt} \]

or \( \vec{\mu} = \frac{q\omega L}{2mc} \) for a current loop.
\[
\hat{\mu} = \frac{eL^2}{2mc}\quad \text{\& for any system of charges, all of which have the same charge/mass ratio}
\]

Thus we get

\[
d\hat{L} = \frac{eB}{2mc} \hat{L} \times \hat{B}
\]

Angular momentum precesses around the magnetic field.

\[
\frac{eB}{2mc}
\]

Called gyromagnetic ratio of the electron's charge.

\[
\omega = \frac{eB}{2mc}
\]

Direction of precession depends on sign of charge.

Now, consider the quantum mechanical case for electrons.

The spin Hamiltonian is also

\[
H = -\hat{\mu} \cdot \hat{B}
\]

and we can get the same precession quantum-mechanically.

Where, for spin-1/2 electrons:

\[
\hat{\mu} = \frac{eB}{2mc} \hat{s}
\]

And for electrons, \( g^* = \text{Electronic charge} = -|e| \)

\[
g^* = \text{Electronic charge} = -|e|
\]

For electrons, gyromagnetic ratio

\[
g^* = \text{Electronic charge} = -|e|
\]

\[
\hat{\mu} = \frac{eB}{2mc} \hat{s}
\]

Where we can also write

\[
\hat{\mu} = \frac{eB}{2mc} \hat{s} = (\sigma_x, \sigma_y, \sigma_z)
\]
or equivalently, for electrons ($q = -e$)

$$\mu = -\frac{e}{2m} g B$$

where $\mu_B = \frac{e}{2m} \frac{\hbar}{2mc} = \text{Bohr magneton}$

(Just as [Note that Bohr magneton for nucleon is \( \sim 2000 \times \) smaller])

$$S = \frac{\hbar}{2}$$

$$\gamma = -\frac{g e B}{2mc} = -\frac{e}{m} \text{ taking } g = 2.$$  

(twice as large as gyromagnetic ratio for orbital motion)

Energy levels. Suppose $B \parallel \hat{z}$

Then

$$H = -\gamma S \cdot B$$

$$H = +\frac{e}{2m} \frac{\hbar}{2mc} \gamma S \cdot B = +\frac{e}{2mc} \gamma B^2 \sigma_z B = \mu_B \sigma_z B$$

Eigenstates are $(0)$ and $(1)$

with eigenvalues $\pm \frac{\hbar}{2mc} \gamma B$

So splitting

$$\omega = \frac{\hbar}{2m} \gamma B$$

where $\omega_{B_0} = \frac{eB}{mc}$
Time-dependent Schrödinger equation:

\[ H|\psi(t)\rangle = \frac{i}{\hbar} \frac{d}{dt} |\psi(t)\rangle \]

Formally, since \( H \) is time-independent,

\[ |\psi(t)\rangle = \exp\left(-i\frac{Ht}{\hbar}\right) |\psi(0)\rangle \]

\[ = \exp\left[ -i\mu_B \vec{B} \cdot \vec{r} \cdot \frac{t}{\hbar} \right] |\psi(0)\rangle \]

for electrons (since \( H = \mu_B \vec{B} \cdot \vec{r} \) for electrons)

Now consider the operator

\[ \exp[\cdot] = \exp\left[i \vec{K} \cdot \vec{\sigma}\right] \]

where \( \vec{K} = \frac{\mu_B \vec{B} t}{\hbar} \)

\[ = \text{const. (i.e. a constant vector which commutes with everything)} \]

I will first note that

\[ (\vec{K} \cdot \vec{\sigma})^2 = K^2 \]

**Proof:**

\[ (\vec{K} \cdot \vec{\sigma})^2 = (K_x \sigma_x + K_y \sigma_y + K_z \sigma_z)^2 \]

\[ = K_x^2 \sigma_x^2 + K_y^2 \sigma_y^2 + K_z^2 \sigma_z^2 + K_x K_y (\sigma_x \sigma_y + \sigma_y \sigma_x) \]

\[ + K_x K_z (\sigma_x \sigma_z + \sigma_z \sigma_x) + K_y K_z (\sigma_y \sigma_z + \sigma_z \sigma_y) \]
But $\delta_{ij}^2 = 1$, $\delta_{ij}^0 + \delta_{ij}^2 = 0$ so

$$(\bar{K} \cdot \bar{\sigma})^2 = K_x^2 + K_y^2 + K_z^2 = K^2 \quad \text{QED}$$

Then $\exp \left[ i(\bar{K} \cdot \bar{\sigma}) \right]$

$$= \sum_{n=0}^{\infty} \frac{1}{n!} i^n (\bar{K} \cdot \bar{\sigma})^n$$

But $(\bar{K} \cdot \bar{\sigma})^n = K^n$ $n$ even

$$= (\bar{K} \cdot \bar{\sigma}) K^{n-1} \quad n \text{ odd}$$

Thus $\exp \left[ i\bar{K} \cdot \bar{\sigma} \right] = \sum_{n=0; 2, 4, \ldots} \frac{1}{n!} K^n$

$$+ \sum_{n=1, 3, 5, \ldots} \frac{i^n}{n!} (\bar{K} \cdot \bar{\sigma}) K^{n-1}$$

$$= 1 - \frac{K^2}{2!} + \frac{K^4}{4!} - \frac{K^6}{6!} + \ldots$$

$$+ i \frac{(\bar{K} \cdot \bar{\sigma})}{|\bar{K}|} \left[ K - \frac{K^3}{3!} + \frac{K^5}{5!} - \ldots \right]$$

$$= \cos K + i K \cdot \bar{\sigma} \sin K$$
For our particular case of interest,

$$\exp \left[ -i \frac{\mu B \cdot \hat{B} t}{\hbar} \right]$$

$$= \cos \left( \frac{\mu B \cdot \hat{B} t}{\hbar} \right) - i \sin \left( \frac{\mu B \cdot \hat{B} t}{\hbar} \right) \left( \frac{\hat{B} \cdot \hat{B}}{\hbar} \right)$$

$$= \left[ \cos \left( \frac{\omega_0 t}{2} \right) - i \sin \left( \frac{\omega_0 t}{2} \right) \left( \frac{\hat{B} \cdot \hat{B}}{\hbar} \right) \right]$$

Therefore

$$|\psi(t)\rangle = \left[ \cos \left( \frac{\omega_0 t}{2} \right) - i \sin \left( \frac{\omega_0 t}{2} \right) \left( \frac{\hat{B} \cdot \hat{B}}{\hbar} \right) \right] |\psi(0)\rangle$$

E.g. suppose, in particular, that $\hat{B} \parallel \hat{z}$.

Then $\hat{B} \cdot \hat{B} = \sigma_\uparrow \sigma_\downarrow = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$.

$$\begin{bmatrix} \cos \left( \frac{\omega_0 t}{2} \right) \\ -i \sin \left( \frac{\omega_0 t}{2} \right) \end{bmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \begin{pmatrix} \exp \left( -i \frac{\omega_0 t}{2} \right) \\ 0 \end{pmatrix} = \begin{pmatrix} \exp \left( -i \omega_0 t \right) \\ 0 \end{pmatrix} = \begin{pmatrix} \exp \left( +i \omega_0 t \right) \\ 0 \end{pmatrix}$$
This is a complete solution of the evolution operator \( U(t) = \exp(-iHt/\hbar) \).

Suppose, e.g., that

\[
|\psi(t)\rangle = \begin{pmatrix} \cos \theta/2 & e^{-i\phi/2} \\ \sin \theta/2 & e^{i\phi/2} \end{pmatrix}
\]

This is an eigenstate of \( \hat{n} \cdot \hat{\mathbf{z}} \) with eigenvalue \( +\frac{n}{2} \) where \( \hat{n} \) is as shown below:

\[ \theta \quad \hat{n} \quad \phi \]

Then

\[
|\psi(t)\rangle = \begin{pmatrix} \cos(\theta/2) \exp(-i(\phi + \omega_0 t)/2) \\ \sin(\theta/2) \exp(+i(\phi + \omega_0 t)/2) \end{pmatrix}
\]

1.e. State precesses around \( \hat{\mathbf{z}} \) axis with frequency \( \omega_0 \), just as in the classical case, where \( \omega_0 = \varepsilon B \).
Notice a strange feature of this state: if we look at the state after time $t = 2\pi/\omega$, the spin has rotated by $2\pi$ about the $z$ axis. However, the resulting state is

$$|\psi(t)\rangle \quad t = 2\pi/\omega = \left( \cos\left(\frac{\pi}{2}\right) e^{-i\phi/2} \right) - \left( \sin\left(\frac{\pi}{2}\right) e^{i\phi/2} \right)$$

i.e. it has rotated into itself!

D. c. + a. c. magnetic field; "rotating frame."

Let us consider

$$H|\psi\rangle = i\hbar \frac{d}{dt} \langle\psi(t)\rangle$$

and now suppose that

$$H = H_{dc} + H_{ac}$$

where $H_{dc} = -\vec{\mu} \cdot \vec{B}_{dc}$

and $H_{ac} = -\vec{\mu} \cdot \vec{B}_{ac}$

where $\vec{B}_{dc} = B_z$

and $\vec{B}_{ac} = \vec{B}_{ac}(b)$

Then $H_{dc} = \mu_B S_z B$ as before where

$$\mu_B = \frac{e\hbar}{2mc}$$
Then we have

\[(H_{dc} + H_{ac}) |\psi(t)\rangle = i \hbar \frac{d}{dt} |\psi(t)\rangle\]

If \(H_{ac}\) were zero, we would have

\[|\psi(t)\rangle = e^{-i H_{dc} t / \hbar} |\psi(0)\rangle\]

Then let us write the solution to the full problem as

\[|\psi(t)\rangle = e^{-i H_{dc} t / \hbar} |\psi(0)\rangle\]

So

\[(H_{dc} + H_{ac}) e^{-i H_{dc} t / \hbar} |\psi'(t)\rangle = i \hbar \frac{d}{dt} \left(e^{-i H_{dc} t / \hbar} |\psi'(t)\rangle\right)\]

\[= i \hbar \left(-i H_{dc} t / \hbar e^{-i H_{dc} t / \hbar} |\psi'(t)\rangle + 14(t) - i H_{dc} t / \hbar \frac{d}{dt} |\psi(t)\rangle\right)\]

1st terms on left and right cancel out so we get

\[H_{ac} e^{-i H_{dc} t / \hbar} |\psi(t)\rangle = i \hbar e^{-i H_{dc} t / \hbar} \frac{d}{dt} |\psi(t)\rangle\]

Left-multiply by \(e^{i H_{dc} t / \hbar}\) to get

\[\hat{H}_{ac} |\psi(t)\rangle = i \hbar \frac{d}{dt} |\psi(t)\rangle\]
where $\hat{H}_{ac}(t) = e^{i\hat{H}_{ac}t/\hbar} H_{ac}(t) e^{-i\hat{H}_{ac}t/\hbar}$

where is the "a.c. hamiltonian in the rotating frame".

If $H_{ac} = 0$, $|\psi(t)\rangle$ is time-independent, corresponding to a uniformly rotating state.

This is a very useful formulation of the behavior of a spin in a d.c. + time-dependent field, and can be used to interpret spin resonance experiments.

As an illustration, suppose

\[ H_{dc} = -\mu B \sigma_z \text{ with } B_{dc} \parallel \hat{z} \]

and $H_{ac} = -\mu B \sigma_x \text{ with } B_{ac} \cos \omega t$.

Then

\[ \hat{H}_{ac} = -\exp \left[ + i \frac{\mu B \sigma_z B_{dc} t}{\hbar} \right] \mu B \sigma_x B_{ac} \cos \omega t \cdot \exp \left[ - i \frac{\mu B \sigma_z B_{dc} t}{\hbar} \right] \]

Consider $\sigma_x = e^{i\lambda \sigma_z} \sigma_x e^{-i\lambda \sigma_z}$

\[ = (\cos \lambda + i \sigma_z \sin \lambda) \sigma_x (\cos \lambda - i \sigma_z \sin \lambda) \]
\[ = \cos^2 \lambda \sigma_x + \sin^2 \lambda \sigma_z \lambda \sigma_x \sigma_z \]
\[ + i (\sigma_z \sigma_x - \sigma_x \sigma_z) \sin \lambda \cos \lambda \]

But \( \sigma_z \sigma_x \sigma_z = -\sigma_x \sigma_z^2 = -\sigma_x \)

Also: \( [\sigma_z, \sigma_x] = 2i \sigma_y \)

So \( \text{rhs} = \cos^2 \lambda \sigma_x + 2i \sin 2\lambda \sigma_y \)

Thus we have
\[ H_{\text{ac}} = \left[ \cos \frac{\mu B B_{\text{dc}} t}{\hbar} \sigma_x - \sin \frac{\mu B B_{\text{dc}} t}{\hbar} \sigma_y \right] (\mu B B_{\text{dc}}) \cos \lambda t \]

Thus, we can actually produce a very simple Hamiltonian for this time-dependent field in the rotating frame.
Similarly, consider \( H_{ac} = -\mu_B \sigma_y B_{ac} \sin \Omega t \).

Then \( H_{ac} = -\exp \left(i \mu_B \sigma_y B_{ac} t / \hbar \right) \mu_B \sigma_x B_{ac} \cos \Omega t \).

Consider \( \exp \left(-i \mu_B \sigma_y B_{ac} t / \hbar \right) \).

\[
\exp (-i \mu_B \sigma_y B_{ac} t / \hbar)
\]

\[
\exp \left(i \lambda \sigma_y \right) \exp \left(-i \lambda \sigma_y \right)
\]

\[
= (\cos \lambda + i \sigma_z \sin \lambda) \sigma_y (\cos \lambda - i \sigma_z \sin \lambda)
\]

\[
= \cos^2 \lambda \sigma_y^2 + \sin^2 \lambda \sigma_z \sigma_y \sigma_z
\]

\[
+ i \left[ \sigma_z, \sigma_y \right] \sin \lambda \cos \lambda
\]

\[
= (\cos \lambda - \sin \lambda) \sigma_y + 2 \sin \lambda \cos \lambda \sigma_x
\]

\[
= 2 \cos \lambda \sigma_y + \sin 2 \lambda \sigma_x
\]

So the \( H_{ac} \) corresponding to this \( H_{ac} \) is

\[
-\mu_B B_{ac} \left[ \sin \Omega t \left( \frac{\mu_B B_{ac} t}{\hbar} \right) \sigma_x + \cos \left( \frac{\mu_B B_{ac} t}{\hbar} \right) \sigma_y \right] \sin \Omega t
\]
Combine the two:

\[
\tilde{H}_{ac} = -\mu_B B_{ac}\left[\cos \omega t \sigma_x - \sin \omega t \sigma_y\right] + i \sin \omega t \left[\sin \omega t \sigma_x + \cos \omega t \sigma_y\right] \]

\[
= -\mu_B B_{ac} \sigma_x \left(\cos \omega t \cos \omega t + \sin \omega t \sin \omega t\right) \sigma_x
\]

\[
+ \left(-\sin \omega t \cos \omega t \cos \omega t \sin \omega t \right) \sigma_y \]

\[
= -\mu_B B_{ac} \sigma_x \cos \left(\omega t - \omega_{0}\right) \sigma_x
\]

\[
+ \sin \left(\omega t - \omega_{0}\right) \sigma_y \]

If, in particular, \(\Omega = \omega_0\), then the perturbation \(\tilde{H}_{ac}\) corresponds to a static field \(\tilde{H}_{ac} = -\mu_B B_{ac} \sigma_x\).

Then the two eigenstates of \(\tilde{H}_{ac}(t)\) are

\[
|\psi(t)\rangle = e^{-i E \Delta t / \hbar} |\psi(0)\rangle
\]

where \(E = \pm \mu_B B_{ac} \sigma_x / 2\).

\(\psi(0)\) represents a state pointing along the \(x\) axis in the rotating frame.
What does this mean?

Well, it means we have solved the full time-dependent problem, because

$$\left| \psi(t) \right> = e^{-iH_{\text{act}}t/\hbar} \left| \psi(0) \right>$$

and

$$\left| \psi(t) \right> = e^{-iH_{\text{act}}t/\hbar} \cdot e^{-iH_{\text{act}}t/\hbar} \left| \psi(0) \right>$$

$$= e^{-iH_{\text{act}}t/\hbar} \cdot e^{-iH_{\text{act}}t/\hbar} \left| \psi(0) \right>$$

Suppose, e.g., \( \left| \psi(0) \right> \) is an eigenstate of \( \sigma_x \) with eigenvalue 1, i.e.

$$\left| \psi(0) \right> = \frac{1}{\sqrt{2}} \begin{pmatrix} 1 \\ 1 \end{pmatrix}$$

This corresponds to \( \phi = 0, \theta = \frac{\pi}{2} \)

Then

$$e^{i\mu B_{\text{ac}} \sigma_x t/\hbar} \left| \psi(0) \right> = e^{i\mu B_{\text{ac}} \sigma_x t/\hbar} \left| \psi(0) \right>$$

and

$$\left| \psi(t) \right> = e^{i\mu B_{\text{ac}} \sigma_x t/\hbar} \begin{pmatrix} e^{-i\omega t/2} \sqrt{2} \\ e^{+i\omega t/2} \sqrt{2} \end{pmatrix}$$

i.e. spin rotates with frequency \( \omega \) about \( z \) axis, just as in static case.
Periodic Table:

We have 2 1s states in a central potential (spin degeneracy).

<table>
<thead>
<tr>
<th>2</th>
<th>2s</th>
</tr>
</thead>
<tbody>
<tr>
<td>6</td>
<td>2p</td>
</tr>
<tr>
<td>2</td>
<td>3s</td>
</tr>
<tr>
<td>6</td>
<td>3p</td>
</tr>
<tr>
<td>10</td>
<td>3d</td>
</tr>
<tr>
<td>2</td>
<td>4s</td>
</tr>
<tr>
<td>6</td>
<td>4p</td>
</tr>
</tbody>
</table>
A problem involving both spin and orbital degrees of freedom

Consider a problem with a "separable" Hamiltonian

\[ H = H_0 + H_S \]

Then \( \psi \rangle = \psi_0 \rangle \otimes \chi_S \rangle \)

orbital part spin part

Example: hydrogen atom

\[ H = H_0 \] no spin part in absence of magnetic field

\[ H_0 = -\frac{\hbar^2}{2m} \nabla^2 + \frac{e^2}{r} \]

in \( x \)-representation

Solutions are of the form

\[ \psi_{nlm} (r, \theta, \phi) \chi_+ (S_z) \]

and \[ \psi_{nlm} (r, \theta, \phi) \chi_- (S_z) \]

or \[ \psi_{nlmms} \]?

Now suppose we have a weak magnetic field \( \vec{B} \parallel \vec{z} \).
The Hamiltonian describing the coupling of the electron to $\vec{B}$ is

$$H' = -\left(\frac{-eB}{2mc}\right) L_z - \left(-\frac{eB}{mc}\right) S_z$$

where $-e$ is the charge of an electron, $m_e$ is its mass, $-\frac{eB}{mc} S_z$ is the spin magnetic moment,

and $-\frac{eB}{2mc} L_z$ is the orbital magnetic moment.

(As we showed in discussing a current loop.)

Therefore $H = H_{\text{coul}} + H'$

where $H_{\text{coul}} = -\frac{\hbar^2}{2m} \nabla^2 + \frac{e^2}{r}$

Hence $H |n l m m_z\rangle = -\frac{1}{n^2} \text{Ry} - \frac{eB}{2mc} m^\frac{1}{2}$

$$-\frac{eB}{mc} m_s \hbar$$

where

$m = 0, \pm 1, \ldots, \pm l$  
$m_s = \pm \frac{1}{2}$

$$= -\left(\frac{1}{n^2}\right) \text{Ry} - \frac{eB}{2mc} \left(m + 2m_s\right) \hbar$$
Example: \( n = 1 \)

\[
\text{Ground state} \quad \begin{array}{c} \uparrow \downarrow \\ \uparrow \downarrow \end{array} \quad \Delta E = \frac{eB \hbar}{m_e c}
\]

On \( E_{n=1} = -1 \text{Ry} \pm \frac{eB \hbar}{2m_e c} \)

\( n = 2 \)

\( B = 0 \) \text{ This is 8-fold degenerate } \quad 2 \text{ 2s levels} \quad 6 \text{ 2p levels} \)

\( B \neq 0 \) \text{ We get } m = -1, 0, 1

Thus we get:

<table>
<thead>
<tr>
<th>( m )</th>
<th>( m_s )</th>
<th>( m + 2m_s )</th>
<th># of states</th>
<th>degeneracy</th>
</tr>
</thead>
<tbody>
<tr>
<td>-1</td>
<td>( \frac{1}{2} )</td>
<td>-1</td>
<td>1</td>
<td>1</td>
</tr>
<tr>
<td>0</td>
<td>( \frac{1}{2} )</td>
<td>1</td>
<td>1</td>
<td>2</td>
</tr>
<tr>
<td>1</td>
<td>( -\frac{1}{2} )</td>
<td>-1</td>
<td>2</td>
<td>(( l = 0 )</td>
</tr>
<tr>
<td>0</td>
<td>( -\frac{1}{2} )</td>
<td>-1</td>
<td>2</td>
<td>(( l = 1 )</td>
</tr>
<tr>
<td>-1</td>
<td>( \frac{1}{2} )</td>
<td>1</td>
<td>1</td>
<td>1</td>
</tr>
</tbody>
</table>

Thus, we have 5 distinct levels from the original 8-fold degenerate state.

This is called the Zeeman effect.

\[ \alpha \to L + S \text{ in the Hamiltonian (which is non-separable)} \]
Stem-Gerlach experiment

Beam of electrons

Permanent magnet

$H = \text{interaction Hamiltonian}$

$= -\hat{\mu} \cdot \vec{B}$

$F = -\vec{\nabla}H = +\vec{\nabla}(\hat{\mu} \cdot \vec{B})$

Suppose that $\vec{B}$ is not homogeneous. Then use vector identity

$\vec{\nabla}(\hat{\mu} \cdot \vec{B}) = (\hat{\mu} \cdot \vec{\nabla})\vec{B} + (\vec{B} \cdot \vec{\nabla})\hat{\mu} + \hat{\mu} \times (\vec{\nabla} \times \vec{B})$

$+ B \times (\vec{\nabla} \times \hat{\mu})$.

But $\hat{\mu}$ is just a constant. Also, $\vec{\nabla} \times \vec{B} = 0$ in the region where the beam moves.

Thus $\vec{\nabla}(\hat{\mu} \cdot \vec{B}) = B \vec{\nabla}(\hat{\mu} \cdot \vec{\nabla})\vec{B}$
Predominant force is in z-direction, which gives

\[ F_z = \rho (\hat{\mathbf{A}} \hat{\mathbf{z}}) \]

Most important part of this is

\[ \mu_z \frac{\partial B_z}{\partial z} \hat{\mathbf{z}} \]

But \( \mu_z \) can have only one of two values for electron, namely

\[ \pm \frac{e \hbar}{2mc} \]

Therefore, resultant force is in one of two directions and has one of only two magnitudes, leading to the result shown below:

- Path is different for + and - spins.
- Since force is different for + and - spins.