**Physics 828: Problem Set I**

**Due Wednesday, January 16 at 11:59 PM**

Note: all problems are worth 10 points unless otherwise stated

1. (20 pts.) In class, I discussed the translation operator, and explained how it can be expressed in terms of the momentum operator $\mathbf{p}$. In this problem, you will consider rotation operators, and show that they can be expressed in terms of components of the *angular* momentum operator $\mathbf{L}$.

   Start with a wave function $\psi$ expressed in spherical coordinates as $\psi(r, \theta, \phi)$. We now consider a rotation about the z axis by an angle $\phi_0$. Let $R_z(\phi_0)$ be the operator which has the following effect on the wave function $\psi$:
$$R_z(\phi_0)\psi(r, \theta, \phi) = \psi(r, \theta, \phi - \phi_0). \tag{1}$$

   (a). By expressing $\psi(r, \theta, \phi - \phi_0)$ in a Taylor series about $\psi(r, \theta, \phi)$, show that
$$\psi(r, \theta, \phi - \phi_0) = \exp\left(-\phi_0 \frac{\partial}{\partial \phi}\right)\psi(r, \theta, \phi), \tag{2}$$
   and hence that
$$R_z(\phi_0) = \exp\left(-\phi_0 \frac{\partial}{\partial \phi}\right). \tag{3}$$
   Hence, show that
$$R_z(\phi_0) = \exp\left(-i\phi_0 L_z/\hbar\right) \tag{4}$$
   where $L_z = -i\hbar(\partial/\partial\phi)$ is the z component of the angular momentum operator as discussed in Shankar, Chapter 12 (and treated last quarter). Hence, $L_z$ can be said to generate rotations around the z axis, just as $p_z$ generates translations along the z axis. Analogous expressions can be obtained for $L_x$ and $L_y$ (you do not need to derive them here, however).

   (b). Hence, explain why these expressions for $R_z$, $R_x$, and $R_y$ show that rotations about different orthogonal axes do not, in general, commute (i. e., it matters which order the rotation operators are applied).

(c). Show that, if the Hamiltonian is a function only of $L^2$, then it commutes with $R_x$, $R_y$, and $R_z$.

2. **Bloch's Theorem.** Consider a one-dimensional Hamiltonian

$$H = -\frac{\hbar^2}{2m}\frac{d^2}{dx^2} + V(x), \tag{5}$$

where $V(x)$ is a periodic function of x with period a, i. e.

$$V(x + na) = V(x) \tag{6}$$

where $n$ is a positive integer and $a$ is a positive constant.

(a). Show that $[H, T(na)] = 0$, where $T(na)$ is a translation operator for a translation by $na$.

(b). Hence, show that the eigenstates of $H$, denoted $\psi(x)$, can always be written as

$$\psi(x + a) = \exp(ika)\psi(x), \tag{7}$$

where $k$ is a real constant. This is known as *Bloch's Theorem.*

3. **Some properties of the parity operator** (20 pts.):

Recall that if the parity operator is denoted $\mathbf{\Pi}$, then it has two types of eigenstates, usually called even and odd eigenstates, which satisfy

$$\mathbf{\Pi}|\psi\rangle = \pm|\psi\rangle, \tag{8}$$

where the + and - signs correspond to even and odd eigenstates. (In class, I used the notation $P$ for the parity operator.)

(a). Show that $\mathbf{\Pi}$ is unitary.

(b). Hence, show that $\mathbf{\Pi}$ is also Hermitean.

(c). Show that, if a system has a time-independent Hamiltonian which commutes with the parity operator, then a state which has definite parity at time $t = 0$ will have the same parity for all later times.

(d). Let us define *even or odd operators* O by the relation

$$\mathbf{\Pi}^\dagger \mathcal{O} \mathbf{\Pi} = \pm\mathcal{O}, \tag{9}$$

where the plus and minus signs refer to even or odd operators. Show that if $\mathcal{O}$ is an even operator, the matrix element

$$\langle \Psi_1 | \mathcal{O} | \Psi_2 \rangle, \tag{10}$$

vanishes if the two states $|\Psi_1\rangle$ and $|\Psi_2\rangle$ have opposite parity (one is odd and the other is even). Also show that, if $\mathcal{O}$ is an odd operator, this matrix element vanishes if both states have the same parity (both states are even or both states are odd.