Chapter 28: Electrical Circuits

In this section of the course we will discuss electrical circuits, e.g. what happens when a battery and one or more resistors are wired together. Soon we will explore circuits that contain not only resistors but also capacitors.

One of the simplest possible circuits is shown on the right. This circuit consists of one resistor, a few wires, and an EMF device.

**EMF (E) stands for Electro-Motive Force**

A battery is an EMF as is a solar cell and an electric generator. Any device that can do work (W) on an electric charge is an EMF. The EMF is defined to be:

\[ E = \frac{dW}{dq} \text{ (units=Volt)} \]

The direction of the current in this circuit follows our rule that current is in the direction that a positive charge would move.

If we know the values of \( R \) and \( E \) we can calculate how much current (I) is flowing. We can use conservation of energy to do this calculation. The energy that the charge gains moving through the EMF is lost going through the resistor.

Energy gained through EMF = \( Edq \)
Energy lost through resistor = Power*dt = \( I^2Rdt \)
\[ Edq = I^2Rdt \Rightarrow EI = I^2R \]
\[ I = \frac{E}{R} \] (what else did you expect?)

In all circuit diagrams we assume wires have zero resistance.

Here “E” is EMF and not electric field!
Single Loop Electrical Circuits

On the previous page we had an example of a circuit with ONE loop. We can do another example of a one loop circuit using a “real battery” as an example. A real battery is an ideal EMF with a resistor in series. The resistor is part of the battery and is often called an “internal resistance”. Our circuit now looks like:

This is a one loop circuit with two resistors and one EMF.

Rather than directly use conservation of energy as in the last example, let’s keep track of the voltage changes as we move clockwise around the circuit from a to b.

\[ E - Ir - IR = 0 \]

It is important to note that the current through \( r \) (the internal resistor) is the same as the current through \( R \). This is because these two resistors are in SERIES with each other.

The current in the circuit is now:

\[ I = \frac{E}{r + R} \]

Thus the current is reduced when we take into account the internal resistance of a battery.

**Example:** Suppose the EMF is a 12V (car) battery and the circuit contains a 1Ω resistor. In a good car battery the internal resistance can be neglected and battery would deliver 12A. However, as the battery deteriorates (or if it is very cold outside) the internal resistance can not be neglected. If the internal resistance became (e.g.) 5Ω then the battery would only deliver 2A! If the 1Ω represented your started motor then your car most likely would not start when the internal resistance reached 5Ω.
Four Rules of Circuit Analysis

Shortly, we will analyze circuits with many components (e.g. > 2 resistors) with the goal of finding out the current or voltage drop across each component. Circuit analysis is greatly simplified if we follow the following four rules:

1) EMF Rule:
As we move across (or through) an EMF device from the (-) terminal to the (+) terminal the change in potential is +E. If we move from (+) to (-) the change in potential is –E.

2) Resistance Rule:
As we move through a resistor in the direction of current the change in potential is –IR. If we move in a direction opposite to the current then the change in potential is +IR.

3) Loop Rule:
The algebraic sign of the potential changes in any COMPLETE traversal of EVERY loop in a circuit is ZERO. This is sometimes known as Kirchoff’s voltage law but is really just conservation of energy.

Using our battery circuit as an example we find: E-Ir-IR=0

4) Junction Rule:
The algebraic sum of all the currents into a junction must be ZERO. Here we assume that currents entering a junction are positive, those leaving are negative. This sometimes known as Kirchoff’s current law but is really just conservation of electric charge.

\[ I_1 - I_2 - I_3 = 0 \]
Circuit Analysis from a Potential Point of View

We can get a picture of what is going on in a circuit loop by keeping track of the potential differences as a charge traverses a complete loop.

In this circuit a positive charge increases its potential going through the ideal battery part of the EMF and correspondingly loses all of this potential going through the two resistors. The smaller the resistance, the smaller the potential drop as it traverses the resistor.

From a physics point of view this is a very good way of looking at this circuit. However, it is not an efficient way to calculate voltage drops or currents in circuit with many elements.
Resistors in Series

Previously we saw that we could define an equivalent capacitance when we had circuits with two or more capacitors. We would like to develop similar formulas for the case where 2 or more resistors are in series.

TWO RESISTORS ARE IN SERIES WHEN THE PATH OF THE CURRENT HAS TO GO THROUGH BOTH RESISTORS.

In this example we have a circuit with 3 resistors in series. Our four rules tell us:

- \( E - I R_1 + IR_2 + IR_3 = 0 \)
- \( E = I(R_1 + R_2 + R_3) \)
- \( R_{eq} = R_1 + R_2 + R_3 \)

The equivalent resistance of resistors in series is the SUM of the individual resistors. Resistors in series combine like capacitors in parallel!

\[
R_{eq} = \sum_{i=1}^{n} R_i
\]

Remember:
The same current flows through each resistor in series.
Resistors in Parallel

Resistors, like capacitors in parallel, all have the same potential (or voltage drop) across them. However, unless the resistors have the same value of resistance they will have different currents flowing through them.

Let’s calculate the equivalent resistance of 3 resistors in parallel.

The battery voltage is $V$ (or $E$).
The total current flowing in the circuit is:

$I = I_1 + I_2 + I_3$

$I = \frac{V_1}{R_1} + \frac{V_2}{R_2} + \frac{V_3}{R_3}$

$I = \frac{V}{R_1} + \frac{V}{R_2} + \frac{V}{R_3}$

$I = V(\frac{1}{R_1} + \frac{1}{R_2} + \frac{1}{R_3}) = \frac{V}{R_{eq}}$

$$\frac{1}{R_{eq}} = \sum_{i=1}^{n} \frac{1}{R_i}$$

Example: assume we have 3 $6\Omega$ resistors in parallel.
Their equivalent resistance is:

$$\frac{1}{R_{eq}} = \frac{1}{6} + \frac{1}{6} + \frac{1}{6} = \frac{3}{6} \Rightarrow R_{eq} = 3\Omega$$
Multi-loop Circuits

In multi-loop circuits it is important to realize which elements are in series & which are in parallel. All four of the circuits below are identical, even though the position of the resistors moves around. In each circuit $R_1$ and $R_2$ are in parallel since both resistors are connected to the EMF identically. The parallel combination of $R_1$ and $R_2$, $R_{12}$, is in series with $R_3$.

![HRW Fig. 28-17](image)

Example: Let $R_1=10\,\Omega$, $R_2=90\,\Omega$, $R_3=1\,\Omega$, and the EMF=10V. Calculate all currents through and voltage drops across each resistor.

note: $R_{12}=9\,\Omega$, $R_{eq}=10\,\Omega$.

a) The total current out of the EMF is: $10V/10\,\Omega=1A$.
b) The current through $R_3$ is the total current out of the EMF since $R_3$ is in series with the EMF: $I_3=1A$ and therefore $V_3=I_3R_3=1V$.
c) From the loop rule there must be 9V across $R_{12}$ and so both $R_1$ and $R_2$ have 9V across them. Finally, $I_1=9V/10\,\Omega=0.9A$, $I_2=9V/90\,\Omega=0.1A$.
Multi-loop Circuits

We can also find the currents and voltage drops in circuit by just using the “Four Rules”. Here we do not explicitly make an equivalent circuit.

As an example let’s find the currents and voltages in this circuit using the “four rules”.

First the current rule (we pick directions for all currents):

At “a” we have: \( I-I_1-I_2=0 \)
At “b” we have: \( I_1+I_2-I_3=0 \)

or \( I=I_3 \)

Next, use the loop rule to find (we have 2 loops):

\( V-I_1R_1-I_3R_3=0 \)
\( V-I_2R_2-I_3R_3=0 \)

At this point we have 3 equations and 3 unknowns (\( I_1, I_2, I_3 \))

\( V-I_1R_1-I_3R_3=0 \)
\( V-I_2R_2-I_3R_3=0 \)
\( I_1+I_2-I_3=0 \)

The standard way to solve this is to eliminate \( I_3 \) in the loop equations and then use algebra or Cramer’s rule (App E, A-10) to solve two equations with two unknowns.

\( I_1R_1+(I_1+I_2)R_3=V \)
\( I_2R_2+(I_1+I_2)R_3=V \)

We now eliminate \( I_2 \) to get one equation in terms of \( I_1 \).
Multi-loop Circuits continued

\[ I_2 = \frac{V - I_1 R_3}{R_2 + R_3} \]

\[ I_1(R_1 + R_3) + \frac{R_3(V - I_1 R_3)}{R_2 + R_3} = V \]

We now have an equation for \( I_1 \) which we can solve with a bit of algebra.

\[ I_1(R_1 + R_3) - \frac{I_1 R_3^2}{R_2 + R_3} = V(1 - \frac{R_3}{R_2 + R_3}) = V \left( \frac{R_2}{R_2 + R_3} \right) \]

\[ I_1 = \frac{VR_2}{(R_1 + R_3)(R_2 + R_3) - R_3^2} = \frac{VR_2}{R_1 R_2 + R_1 R_3 + R_3 R_2} = \frac{V}{R_1 + R_3 + \frac{R_1 R_3}{R_2}} \]

\[ I_1 = \frac{V}{R_1 + R_3 + \frac{R_1 R_3}{R_2}} = \frac{10V}{10\Omega + 1\Omega + (10\Omega)(1\Omega) / 90\Omega} = 0.9A \]

Note: If a current is negative then you picked the wrong direction for the current.

So, after a lot of algebra we get the same result as before!

So, why do we use this technique if there is so much algebra involved? It turns out that the “four rules” can implemented and solved by a computer (or good calculator) in a very general way that does not require equivalent circuits. Also once you have circuits with many loops the equivalent circuit method also requires a lot of algebra therefore loses any advantage it might have.
Multi-loop, Multi-EMF Circuits

Having an EMF inside a loop usually means that we can not just use the equivalent circuit approach to find all the currents and voltages drops in the circuit. We can however always use our “4 rules” to analyze these circuits.

Example: Here’s a problem with 3 EMFs!

Let’s find the current through each resistor and voltage drop.

1) First, label the circuit (a, b, c, etc) at various branch points.
2) Label currents. Pick a direction for a current.
   If you pick the wrong direction you will wind up with a negative value for the current. That’s ok!
3) Pick a junction in the circuit where 3 currents meet.
   At d) we have $I_1 + I_2 + I_3 = 0$ (Eq. 1) (we could have used c instead)
   (At least one of these currents must be in the wrong direction since 3 positive numbers can’t add to 0!)
4) Pick a loop. Let’s use abdca (a loop gets us back to where we started)
   $+V_1 - I_1 R_1 - V_2 + I_2 R_2 = 0$ (Eq. 2) (its $-V_2$ and $+I_2 R_2$ because of our convention)

At this point in the problem we pause and ask how many equations do we need?
To answer this we need to know how many unknowns there are! At first glance you might say there are 4 unknown voltages and 4 unknown currents. But really there are only 3 unknowns. First realize that $I_4 = I_3$. That leaves 7 unknowns. Second, using Ohm’s law $V_4 = I_4 R_4 = I_3 R_4$ we eliminate another unknown $V$. Again, we could use Ohm’s law for the voltage drop across the 3 other resistors IF we knew their currents. So, the 3 unknowns are: $I_1$, $I_2$, and $I_3$.
Therefore we need 3 independent (and linear) equations to solve for the 3 unknowns.
5) Pick another loop. Let’s use abdfeca (a loop gets us back to where we started)
   $+V_1 - I_1 R_1 + I_3 R_3 + I_3 R_4 - V_3 = 0$ (Eq. 3) (its $-V_3$ and $+I_3 R_3$ and $+I_3 R_4$ because of our convention)
   
At this point we have 3 equations, 3 unknowns and all the I’s and V’s can be calculated!
Assume the following values:

\[ V_1 = 5V, \ V_2 = 10V, \ V_3 = 2V \]
\[ R_1 = 10\Omega, \ R_2 = 20\Omega, \ R_3 = 5\Omega, \ R_4 = 2\Omega. \]

From the previous page our 3 equations are:

\[ I_1 + I_2 + I_3 = 0 \]
\[ +V_1 - I_1R_1 - V_2 + I_2R_2 = (5V) - I_1(10\Omega) - 10V + I_2(20\Omega) = 0 \]
\[ +V_1 - I_1R_1 + I_3(R_3 + R_4) - V_3 = (5V) - I_1(10\Omega) + I_3(7\Omega) - 2V = 0 \]

To solve 3 equations with 3 unknowns takes quite a bit of algebra (which is not the point of this course). I used a computer program (MATHEMATICA) to solve these equations for the 3 currents.

The results are:

\[ I_1 = \frac{5}{82}A \quad I_2 = \frac{23}{82}A \quad I_3 = \frac{-28}{82}A \]

Since \( I_3 \) is negative, its true direction is opposite to what I have drawn above. The same is true for \( I_4 \).

We can find the voltage drops across the resistors using Ohm’s law.

\[ V_1 = \frac{25}{41}V \quad V_2 = \frac{230}{41}V \quad V_3 = \frac{-70}{41}V \quad V_4 = \frac{-28}{41}V \]

The MATHEMATICA program I used is:

\[
m = \{\{-10,20,0\},\{-10,0,7\},\{1,1,1\}\}\\
v = \{5,-3,0\}\\
s = \text{LinearSolve}[m,v]
\]
Multi-loop, Multi-EMF Circuits

Without a computer (or programmable calculator) finding numerical values in most multi-loop circuits is too tedious for a P132 quiz or exam. However, you should be able to write the correct loop equations and current junction equations. Here are two circuits to practice on. The answers are at the end of this chapter’s notes.

Write the loop equation for the loop abdfeca.
Write the current equation at junction c)

Write the loop equation for the loop cdfec.
Circuits With Resistors and Capacitors

The behavior of circuits with resistors and capacitors is more complex than circuits with just resistors or capacitors. As we shall see, in circuits with Rs and Cs currents and voltages are not constant in time. Let’s analyze the following circuit with 1 EMF, 1 resistor and 1 capacitor.

Let’s analyze this circuit using the “loop rule” going clockwise around the circuit. Initially, there is no charge on the capacitor and no current through the resistor.

**Let’s now connect the switch to the battery (position a).**

In order to analyze this circuit we need to update the “4 rules” to include capacitors. We add a “capacitance rule”.

**Capacitance Rule:**

As we move through a capacitor in the direction of +charge to –charge the potential difference is:

\[-V_c = -\frac{q}{C}\]

Going around the circuit in a complete loop we find:

\[V - V_R - V_C = 0\]
\[V - IR - \frac{q}{C} = 0\]

We now remember the relationship between current and charge (I=dq/dt):

\[V - R \frac{dq(t)}{dt} - \frac{q(t)}{C} = 0\]

This is our first example of a differential equation. The solution to the equation is:

\[q(t) = CV \left(1 - e^{-t/RC}\right)\]

We can show that this is a solution by plugging it back into the loop equation.

\[Rdq/dt+q/C=R(V/R)e^{-t/RC}+ (1/C)(CV)(1-e^{-t/RC})=V\]
Charging a Capacitor

We can now find the voltages and currents as a function of time:

\[ V_C = \frac{q(t)}{C} = V \left(1 - e^{-t/RC}\right) \]

\[ V_R = RI(t) = R \frac{dq(t)}{dt} = RCV \frac{d}{dt} \left(1 - e^{-t/RC}\right) = RCV \frac{e^{-t/RC}}{RC} = Ve^{-t/RC} \]

Note, at any instant in time \( V_C + V_R = V \)

\[ V_C + V_R = V \left(1 - e^{-t/RC}\right) + Ve^{-t/RC} = V \]

Finally, the current in the circuit is just \( V_R/R \):

\[ I(t) = \frac{Ve^{-t/RC}}{R} \]

RC is called the “time constant” Its value determines how fast the capacitor charges up (or down) and how fast the current dies off.

The charge on the capacitor starts at zero and builds up exponentially to \( V_C \).

The rate of the build up is determined by \( RC \), the “time constant”.

The capacitor is “charging up” to \( V \).
Discharging a Capacitor

Let’s assume that the capacitor has been charged up to V. Suppose we now flip the switch from a) to b). What happens to \( V_C \) and \( V_R \)?

Again we go around a complete loop and write down the voltage drops across R and C. The big difference now is that the EMF is no longer in the loop.

\[-V_C - V_R = 0\]
\[q(t)/C + IR = 0\]

\[R \frac{dq(t)}{dt} + \frac{q(t)}{C} = 0\]

The solution to this differential equation is (\( q(0) \) is the charge on C before the switch is thrown):

\[q(t) = CV e^{-t/RC} = q(0)e^{-t/RC}\]

To see that this is a solution just plug back into the equation!

So, the voltage on the capacitor decreases exponentially with time:

\[V_C(t) = q(t)/C =Ve^{-t/RC}\]

The voltage across the resistor, \( V_R \), is very interesting! It starts out negative and increases to zero.

\[V_R(t) = R \frac{dq(t)}{dt} = RCV \frac{d}{dt} e^{-t/RC} = -Ve^{-t/RC}\]

Note: at every point in time \( V_C + V_R = 0 \), as it should be since there is no EMF in the circuit.
## Comparing Resistors and Capacitors

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<tr>
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<th><strong>Series</strong></th>
<th><strong>Parallel</strong></th>
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<tbody>
<tr>
<td><strong>Capacitor</strong></td>
<td>[ \frac{1}{C_{eq}} = \sum_{i=1}^{n} \frac{1}{C_i} ]</td>
<td>[ C_{eq} = \sum_{i=1}^{n} C_i ]</td>
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<tr>
<td>What’s the same?</td>
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<td><strong>Resistor</strong></td>
<td>[ R_{eq} = \sum_{i=1}^{n} R_i ]</td>
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</tbody>
</table>
Write the loop equation for the loop abdfeca.
$I_3=I_4=I_5$ since these resistors are in series.
$+V_1-I_1R_1+I_3R_3+I_3R_4-V_3+I_3R_5=0$

Write the current equation at junction c)
$-I_1-I_2-I_5=0$ or $-I_1-I_2-I_3=0$ (negative by convention)

Write the loop equation for the loop cdfec.
Again, $I_3=I_4=I_5$ since these resistors are in series.
$-I_2R_2+V_2+I_3R_3+V_3+I_3R_4-V_4+I_3R_5=0$