For a free particle, $\hat{H} = \frac{\hat{p}^2}{2m}$. At $t = 0$, $\psi(x) = A x \exp[-\frac{x^2}{2\ell^2}]$. Note that $\int_{-\infty}^{\infty} dx \ x^2 \exp[-a \ x^2] = \sqrt{\pi/(2a^3/2)}$. All integrals that you encounter can be completed using the table provided, but this is a difficult calculation and integral evaluation should be saved until you have finished all other problems. You will receive all of the credit for this problem by setting up the correct integrals in part (a), but you should be able to evaluate the integrals in part (b). You can shift arguments by imaginary amounts (e.g., $x$ goes to $x + iy$) to obtain simple exponents as long as you do not move the contour of integration through a pole or cut, and these are not encountered here.

(1a) What is $\psi(x, t)$?

$$\langle x | \psi(t) \rangle = A x \exp \left[-\frac{x^2}{2\ell^2} \right]$$

$$1 = A^2 \int_{-\infty}^{\infty} dx \ x^2 e^{-2x^2/\ell^2} = A^2 \frac{\sqrt{\pi}}{2(2\ell^2)^{3/2}} = A^2 \frac{\sqrt{\pi} \ L^3}{2^{3/2}} = 1$$

$$A = \sqrt{\frac{2}{}\sqrt{\pi} L^3}$$

$$\langle x | \psi(t) \rangle = \langle x | e^{-i \hat{H} t} | \psi(\infty) \rangle = \int dp \int dy \ \langle x | p \rangle \langle p | e^{-i \hat{H} t} | y \rangle \langle y | \psi(\infty) \rangle$$

$$\psi(x, t) = \int dp \int dy \ \frac{e^{ipx}}{\sqrt{2\pi}} \ \exp \left[-i \frac{p^2}{2m} t \right] \ \frac{e^{-iy^2}}{\sqrt{2\pi}} \ A y \exp \left[-\frac{y^2}{2\ell^2} \right]$$

These integrals can be completed easily but this form, an expansion in terms of eigenstates of $\hat{H}$, receives full credit.
(1b) What are \( \langle \hat{P} \rangle (t) \) and \( \langle \hat{H} \rangle (t) \)? You should be able to completely evaluate any integrals in this part. Use Ehrenfest's principle.

Since \( \frac{\partial \hat{H}}{\partial t} = 0 \), and \( [\hat{P}, \hat{H}] = [\hat{H}, \hat{H}] = 0 \), neither of these depends on time. Evaluate using \( \psi (t=0) \).

\[
\langle \hat{P} \rangle = \int_{-\infty}^{\infty} dx \: \psi(x, t=0) \left( -i \frac{\partial}{\partial x} \right) \psi(x, t=0)
\]

\[
= 0 \quad \text{if } \psi \text{ odd}, \quad \frac{\partial \psi}{\partial x} \text{ is even}
\]

\[
\langle \hat{H} \rangle = \int_{-\infty}^{\infty} dx \: \psi(x, t=0) \left[ -\frac{1}{2m} \frac{\partial^2}{\partial x^2} \right] \psi(x, t=0)
\]

\[
= \frac{2}{\sqrt{2\pi} \mu} \int_{-\infty}^{\infty} dx \: x^2 e^{-x^2 / 2L^2} = \frac{2}{\sqrt{2\pi} \mu} \left[ -\frac{x^2}{L^2} e^{-x^2 / 2L^2} - 2 \frac{x}{L^2} e^{-x^2 / 2L^2} + \frac{2}{L^4} e^{-x^2 / 2L^2} \right]
\]

\[
= -\frac{2}{L^2} e^{-x^2 / 2L^2} - \frac{4x}{L^2} e^{-x^2 / 2L^2} - \frac{4x^3}{L^4} e^{-x^2 / 2L^2}
\]

\[
\langle \hat{H} \rangle = \frac{\hat{A}^2}{2m} \int_{-\infty}^{\infty} dx \left( \frac{x^2}{L^2} + 4 \frac{x^4}{L^4} \right) e^{-x^2 / 2L^2}
\]

\[
= \frac{\hat{A}^2}{2m} \left[ \frac{6}{L^2} \left( -\frac{\partial}{\partial a} \right) + \frac{4}{L^4} \frac{\partial^2}{\partial a^2} \right] \int_{-\infty}^{\infty} dx \: e^{-a x^2} \left| _{a = \frac{2}{L^2}} \right.
\]

\[
= \frac{\hat{A}^2}{2m} \left[ \frac{6}{L^2} \left( \frac{1}{2} a - \frac{3}{2} \right) + \frac{4}{L^4} \frac{3}{4} a \right] \left[ 4 \pi a^{-\frac{3}{2}} \right] \left| _{a = \frac{2}{L^2}} \right.
\]

\[
= \frac{\sqrt{\pi} \hat{A}^2}{2m} \left[ \frac{6}{L^2} \left( \frac{1}{2} \frac{2}{L^2} - \frac{3}{2} \right) + \frac{4}{L^4} \left( \frac{3}{4} \left( \frac{2}{L^2} \right) \right) \right] \left| _{a = \frac{2}{L^2}} \right.
\]

\[
= \frac{9}{2m L^2}
\]
At \( t = 0 \) a particle in a harmonic oscillator potential is placed in a state 
\( |\psi(t = 0)\rangle = \frac{1}{\sqrt{2}}|\phi_2\rangle - \frac{1}{\sqrt{2}}|\phi_0\rangle \), where \( |\phi_n\rangle \) are eigenstates of the 
harmonic oscillator Hamiltonian, \( \hat{H} = \frac{\hat{p}^2}{2m} + \frac{K}{2} \hat{x}^2 \).

(20) \hspace{1cm} (2a) What is \( |\psi(t)\rangle \)? Compute \( \langle \hat{X} \rangle (t) \) and \( \langle \hat{P} \rangle (t) \). Define any 
constants you use.

\[
|\psi(t)\rangle = \frac{1}{\sqrt{2}} \exp(-iE_2t)|\phi_2\rangle - \frac{1}{\sqrt{2}} \exp(-iE_0t)|\phi_0\rangle
\]

where \( E_n = (n+\frac{1}{2})\omega_0 \), \( \omega_0 = \sqrt{\frac{K}{m}} \)

\[
\hat{X} = \frac{1}{\sqrt{2}\beta} (\hat{a} + \hat{a}^+) \quad \hat{P} = \frac{\beta}{i\sqrt{2}} (\hat{a} - \hat{a}^+)
\]

\[
\langle \hat{X} \rangle = 0 \quad \text{\{ \( \hat{a} \) and \( \hat{a}^+ \) have non-vanishing matrix elements only between 
states with \( n \) differing by 1 \}}
\]
\[
\langle \hat{P} \rangle = 0
\]
(2b) Compute $\langle \hat{X}^2 \rangle (t)$ and $\langle \hat{P}^2 \rangle (t)$.

\[ \hat{X}^2 = \frac{1}{2\beta^2} (\hat{a}^+ \hat{a}^+ + \hat{a} \hat{a} = \frac{1}{2\beta^2} (\hat{a}^+ \hat{a}^+ + \hat{a} \hat{a} + \hat{a} \hat{a} + \hat{a}^+ \hat{a}) \]

\[ = \frac{1}{2\beta^2} (\hat{a}^2 + \hat{a}^2 + 2N + 1) \]

\[ \hat{P}^2 = -\frac{\beta^2}{2} (\hat{a}^+ \hat{a} - \hat{a} \hat{a}^+) = -\frac{\beta^2}{2} (\hat{a}^2 - \hat{a} \hat{a}^+ - \hat{a}^+ \hat{a} + \hat{a}^2) \]

\[ = -\frac{\beta^2}{2} (\hat{a}^2 + \hat{a}^2 - 2N - 1) \]

\[ N = \hat{a} \hat{a}^+ = N + 1 \]

\[ \langle \hat{a}^2 \rangle = \left( -\frac{1}{\sqrt{2}} \langle \phi_0 | e^{iE_0 t} \hat{a} \left( \frac{1}{\sqrt{2}} e^{-iE_0 t} | \phi_2 \rangle \right) \right) \]

\[ \left( \langle \phi_0 | \hat{a}^2 | \phi_2 \rangle = \sqrt{2} \right) \]

\[ \left( \langle \hat{a}^2 \rangle = -\frac{1}{\sqrt{2}} e^{-i(E_2 - E_0)t} = -\frac{1}{\sqrt{2}} e^{-2i\omega_0 t} \right) \]

\[ \langle \hat{a}^2 \rangle^* = \langle \hat{a}^2 \rangle = -\frac{1}{\sqrt{2}} e^{2i\omega_0 t} \]

\[ \langle 2N + 1 \rangle = \langle \psi(t) | \left[ \frac{5}{\sqrt{2}} e^{-iE_0 t} | \phi_2 \rangle - \frac{1}{\sqrt{2}} e^{-iE_0 t} | \phi_0 \rangle \right] \]

\[ = \frac{5}{2} + \frac{1}{2} = 3 \]

\[ \langle \hat{X}^2 \rangle = \frac{1}{2\beta^2} \left( 3 - \sqrt{2} \cos (2\omega_0 t) \right) = \beta^2 \left( \frac{3}{2} - \frac{1}{\sqrt{2}} \cos (2\omega_0 t) \right) \]

\[ \langle \hat{P}^2 \rangle = -\frac{\beta^2}{2} \left( -\sqrt{2} \cos (2\omega_0 t) \right) = \beta^2 \left( \frac{3}{2} + \frac{1}{\sqrt{2}} \cos (2\omega_0 t) \right) \]
3) There are three bound states in the potential well shown below. \( V(x) = 0 \) for \( x < -a \), \( V(x) = -V \) for \(-a < x < a\), and \( V(x) = \infty \) for \( x > a \). The third bound state is barely bound, with binding energy very near zero. Find the wave function in each region and set up the boundary conditions. Specify any constants you use in terms of \( m \), \( a \) and \( V \). Use the boundary conditions to approximate \( V \) in terms of \( m \) and \( a \), given the fact that there are only three bound states and the third is barely bound. Sketch the three bound states.

\[
\psi(x) = \begin{cases} 
A e^{kx}, & x \leq -a \\
B \sin[k(x-a)], & -a < x < a \\
0, & x \geq a 
\end{cases}
\]

\[
E = -\frac{k^2}{2m}
\]

\[
\psi(a) = 0 \text{ is imposed by choosing } \sin[k(x-a)].
\]

\[
\psi(-a) = A e^{-ka} = -B \sin(2ka)
\]

\[
\psi(-a) = KA e^{-ka} = kB \cos(2ka)
\]

\[
-2ka \cot(2ka) = 2kA \]

\[
3\text{rd bd. state just bound}
\]

\[
8mV \frac{a^2}{2} \geq \left( \frac{5\pi}{2} \right)^2
\]

\[
V \geq \frac{25\pi^2}{32ma^2}
\]