1) The wave function for a particle is sketched below. Assume $\Delta x \ll a$.

(a) What is $\langle \hat{X} \rangle$?

$$\langle \hat{X} \rangle = \frac{|1|^2 \times (-2a) + |3|^2 \times (-a) + |2|^2 \times (2a)}{|1|^2 + |3|^2 + |2|^2} = \frac{-3a}{14} = -\frac{3}{14} a$$

(b) List possible values of position that can be measured and their probability.

<table>
<thead>
<tr>
<th>possible $x$</th>
<th>$\varphi(x)$</th>
</tr>
</thead>
<tbody>
<tr>
<td>$-2a$</td>
<td>$1/4$</td>
</tr>
<tr>
<td>$-a$</td>
<td>$3/14$</td>
</tr>
<tr>
<td>$2a$</td>
<td>$1/4$</td>
</tr>
</tbody>
</table>
2) A particle in a box, $0 < x < L$, is in the state:
\[ \psi(x) = 2\phi_2(x) + 4\phi_4(x) - \phi_5(x). \]

(a) What is the lowest energy that can be measured? Assume mass $m$ and give the answer in terms of $m$ and other constants. Define any constants you use.

\[ E_2 = \frac{k^2}{2m} = \frac{1}{2m} \left( \frac{2\pi}{L} \right)^2 = \frac{2\pi^2}{mL^2}. \]

(b) What is $\langle \hat{X} \rangle$? Give an answer correct at any time $t$.

Fix norm.

\[ \psi(x,t) = \frac{2}{\sqrt{21}} e^{-\frac{i\omega_1 t}{2}} \phi_2(x) + \frac{4}{\sqrt{21}} e^{-\frac{i\omega_4 t}{2}} \phi_4(x) - \frac{1}{\sqrt{21}} e^{-i\omega_5 t} \phi_5(x) \]

\[ \phi_n(x) = \sqrt{\frac{2}{L}} \sin \left( \frac{n\pi x}{L} \right), \quad \omega_n = \left( \frac{\pi^2}{2mL^2} \right)^n \]

\[ \langle \hat{X} \rangle = \int dx \; |\psi(x,t)|^2 \]

\[ = \int dx \; \left\{ \frac{4}{21} \phi_2^2(x) + \frac{16}{21} \phi_4^2(x) + \frac{1}{21} \phi_5^2(x) \right. \]

\[ + \frac{16}{21} \phi_2(x) \phi_4(x) \cos \left[ (\omega_4 - \omega_4) t \right] - \text{vanishes after integration - parity} \]

\[ - \frac{4}{21} \phi_2(x) \phi_5(x) \cos \left[ (\omega_2 - \omega_5) t \right] \]

\[ - \frac{8}{21} \phi_4(x) \phi_5(x) \cos \left[ (\omega_4 - \omega_5) t \right] \]

This receives full credit.
You did not need to work out the details (as I mentioned in class) and actually complete the integrals. To do so:

\[ dx \times \phi_n^2(x) = \frac{1}{2} \quad \text{every eigenstate is centered at the middle of the box} \]

\[ dx \times \phi_2(x) \phi_4(x) = 0 \quad \text{integrand antisymmetric about middle of box} \]

\[ dx \times \phi_2(x) \phi_5(x) = -\frac{80}{441 \pi^2} L \]

\[ dx \times \phi_4(x) \phi_6(x) = -\frac{160}{81 \pi^2} L \]

\[ \langle x \rangle^2 = \frac{L}{2} + \frac{4}{21} \frac{80}{441 \pi^2} L \cos \left[ \frac{\pi^2}{2m^2} (4-25) \right] \]

\[ + \frac{8}{21} \frac{160}{81 \pi^2} L \cos \left[ \frac{\pi^2}{2m^2} (16-25) \right] \]

\[ \approx L \left\{ \frac{1}{2} + \frac{320}{9261 \pi^2} \cos \left( \frac{21 \pi^2}{2mL^2} t \right) \right\} \]

\[ + \frac{1280}{1701 \pi^2} \cos \left( \frac{9 \pi^2}{2mL^2} t \right) \]
3) Assume $\hat{H}\phi_n = E_n\phi_n$. At $t = 0$, $|\psi> = \frac{1}{\sqrt{6}}|\phi_1> + \frac{2}{\sqrt{6}}|\phi_3>$. 

(a) What is the probability to find the particle with $0 < x < L$ at time $t = \pi/E_1$. Use $<x|\phi_n> = \phi_n(x)$.

\[ \mathcal{P}(0 < x < L) = \int_0^L \left( \frac{1}{\sqrt{6}} e^{-iE_1 t} \phi_1(x) + \frac{2}{\sqrt{6}} e^{-iE_3 t} \phi_3(x) \right)^2 dx \]

You do not know the eigenfunctions, so this is as far as you can go.

(b) The particle is measured to have energy $E_3$. What is the probability to find $0 < x < L$ at a time $t = \pi/E_1$ later?

\[ \text{measure } E_3 \Rightarrow |\psi(t=0)> = |\phi_3> \quad \left\{ \begin{array}{l} \text{the new state is the eigenstate} \\ \text{even an eigenstate has a simple time-dependence, but this is not needed} \end{array} \right. \]

\[ |\psi(t)> = e^{-iE_3 t} |\phi_3> \]

\[ \mathcal{P}(0 < x < L) = \int_0^L |\phi_3(x)|^2 dx \]

- there is no time dependence.
4) For a free particle, $\hat{H} = \frac{p^2}{2m}$. At $t = 0$, $\psi(x) = A \sin(\frac{2\pi x}{L})$ for $-L < x < L$, $\psi(x) = 0$ for $|x| > L$.
(a) What is $\psi(x, t)$?

Need $\langle p | \psi \rangle = \int_{-L}^{L} dx \langle x | \psi \rangle \langle x | p \rangle$

$$= \sqrt{\frac{1}{2\pi L}} \int_{-L}^{L} dx \ e^{-i \frac{p}{2m} x} \sin \left( \frac{2\pi x}{L} \right)$$

Use $\int_{-L}^{L} dx \cos(\pi x) \sin \left( \frac{2\pi x}{L} \right) = 0$ (generic for free particle)

$$\psi(x, t) = -\frac{2}{\sqrt{L}} \int_{-\infty}^{\infty} dp \ \frac{\sin \left( \frac{4\pi p}{L} \right)}{L^2 p^2 + \frac{\pi^2}{4}} \ e^{i p x} \ e^{-i \frac{p^2 t}{2m}}$$

(b) What are $\langle \hat{P} \rangle$ and $\langle \hat{H} \rangle$?

Neither of these depends on time.

$\langle \hat{P} \rangle = 0$ by symmetry; $\langle p | \psi \rangle$ is even and $\int dp \ p \ |\psi(p)|^2$ has an odd integrand,

or compute $\int_{-L}^{L} dx \ \psi^* (x) \left( -i \frac{\partial}{\partial x} + A(x) \right) \psi(x)$.

$$\langle \hat{H} \rangle = -\frac{1}{2m} \int_{-L}^{L} dx \ \psi^* (x) \frac{\partial^2 \psi(x)}{\partial x^2} = \left( \frac{1}{2m} \right) \left( \frac{2\pi}{L} \right)^2 \langle \psi | \psi \rangle$$

$$\langle \hat{H} \rangle = \frac{2\pi^2}{m L^2}$$

~ more complicated expressions result in momentum representation